EXISTENCE AND UNIQUENESS OF SOLUTIONS TO A NON-LOCAL EQUATION WITH MONOSTABLE NONLINEARITY

JÉRÔME COVILLE[2,3], JUAN DÁVILA[1,2], AND SALOMÉ MARTÍNEZ[1,2]

Abstract. Let $J \in C(\mathbb{R})$, $J \geq 0$, $\int_{\mathbb{R}} J = 1$ and consider the nonlocal diffusion operator $\mathcal{M}[u] = J \star u - u$. We study the equation

$$\mathcal{M}u + f(x,u) = 0, \quad u \geq 0 \quad \text{in } \mathbb{R},$$

where f is a KPP type non-linearity, periodic in x. We show that the principal eigenvalue of the linearization around zero is well defined and a that a non-trivial solution of the nonlinear problem exists if and only if this eigenvalue is negative. We prove that if, additionally, J is symmetric then the non-trivial solution is unique.

1. Introduction

Reaction-diffusion equations have been used to describe a variety of phenomena in combustion theory, bacterial growth, nerve propagation, epidemiology and spatial ecology [13, 12, 15, 19]. However, in many situations such as in population ecology, dispersal is better described as a long range process rather than as a local one, and integral operators appear as a natural choice. Let us mention in particular the seminal work of Kolmogorov, Petrovskii, Piskounov [16] (1937), where the authors introduced a model for the dispersion of gene fractions involving a non-local linear operator and a non-linearity of the form $u(1-u)$, which now many authors call a KPP type nonlinearity.

Nonlocal dispersal operators usually take the form $\mathcal{M}[u] = \int_{\mathbb{R}^N} k(x,y)u(y)dy - u(x)$ where $k \geq 0$ and $\int_{\mathbb{R}^N} k(y,x)dy = 1$ for all $x \in \mathbb{R}^N$. They have been mainly used in discrete time models [17] while continuous time versions have also been recently considered in population dynamics [14, 18]. Steady state and travelling wave solutions for single equations have been studied in the case $k(x,y) = J(x-y)$ with J even, for some specific reaction nonlinearities in [1, 10, 8, 2].

In this work we restrict ourselves to 1 dimension and take

$$k(x,y) = J(x-y).$$

We are interested in the existence/non-existence and uniqueness of solutions of the following problem

$$\mathcal{M}[u] + f(x,u) = 0 \quad \text{in } \mathbb{R}, \tag{1.1}$$

where $f(x,u)$ is a KPP type nonlinearity, periodic in x, and

$$\mathcal{M}[u] := J \star u - u. \tag{1.2}$$

Date: October 31, 2006.

We assume that J satisfies

$$J \in C(\mathbb{R}) \quad J \geq 0, \quad \int_{\mathbb{R}} J = 1, \tag{1.3}$$

$$\text{there exist } a < 0 < b \text{ such that } J(a) > 0,\ J(b) > 0. \tag{1.4}$$

On f we assume that:

$$\begin{cases} f \in C(\mathbb{R} \times [0, \infty)) \text{ and is differentiable with respect to } u \\ \text{for each } u\ f(\cdot, u) \text{ is periodic with period } 2R \\ f_u(\cdot, 0) \text{ is Lipschitz} \\ f(\cdot, 0) \equiv 0 \text{ and } f(x, u)/u \text{ is decreasing with respect to } u \\ \text{there exists } M > 0 \text{ such that } f(x, u) \leq 0 \text{ for all } u \geq M \text{ and all } x. \end{cases} \tag{1.5}$$

The model example of such a nonlinearity is

$$f(x, u) = u(a(x) - u)$$

where $a(x)$ is periodic and Lipschitz.

In a recent work, Berestycki, Hamel and Roques [2] studied the analog of (1.1) with a divergence operator in a periodic setting. They showed existence of nontrivial solutions provided the linearization of the equation around 0 has a negative first periodic eigenvalue.

We prove the following result

Theorem 1.1. *Assume J satisfies* (1.3), (1.4) *and f satisfies* (1.5). *Then there exists a non-trivial, continuous, periodic solution of* (1.1) *if and only if*

$$\lambda_1(\mathcal{M} + f_u(x, 0)) < 0,$$

where λ_1 is the principal eigenvalue of the linear operator $-(\mathcal{M} + f_u(x, 0))$ in the set of $2R$-periodic continuous functions. Moreover if $\lambda_1 \geq 0$ then any uniformly continuous, nonnegative bounded solution is identically zero.

To prove Theorem 1.1, we first need to show that the principal periodic eigenvalue of $-(\mathcal{M} + f_u(x, 0))$ is well defined. Let us introduce some notation:

$$C_{per}(\mathbb{R}) = \{u : \mathbb{R} \to \mathbb{R} \mid u \text{ is continuous and } 2R\text{-periodic}\}$$

$$C^{0,1}_{per}(\mathbb{R}) = \{u : \mathbb{R} \to \mathbb{R} \mid u \text{ is Lipschitz and } 2R\text{-periodic}\}$$

Theorem 1.2. *Suppose $a(x) \in C^{0,1}_{per}(\mathbb{R})$. Then the operator $-(\mathcal{M} + a(x))$ has a unique principal eigenvalue λ_1 in $C_{per}(\mathbb{R})$, that is, there is a unique $\lambda_1 \in \mathbb{R}$ such that*

$$\mathcal{M}[\phi_1] + a(x)\phi_1 = -\lambda_1 \phi_1 \quad \textit{in } \mathbb{R} \tag{1.6}$$

admits a positive solution $\phi_1 \in C_{per}(\mathbb{R})$. Moreover λ_1 is simple, that is, the space of $C_{per}(\mathbb{R})$ solutions to (1.6) *is one dimensional.*

In [2] the authors prove that for divergence operators the periodic solution is unique in the class of bounded nonnegative solutions. A similar result is true for the non-local problem (1.1), but this time we need J to be symmetric, that is:

$$J(x) = J(-x) \quad \text{for all } x \in \mathbb{R}. \tag{1.7}$$

Note however that for the existence result, Theorem 1.1, we do not need this condition.

Theorem 1.3. *Assume J satisfies (1.3), (1.4), (1.7), f satisfies (1.5). Let u be a non-negative, bounded, uniformly continuous solution u to (1.1) and let λ_1 be the principal eigenvalue of the operator $-(\mathcal{M} + f_u(x, 0))$ with periodic boundary conditions.*

a) If $\lambda_1 < 0$ then either $u \equiv 0$ or $u \equiv p$, where p is the positive periodic solution of Theorem 1.1.

b) If $\lambda_1 \geq 0$ then $u \equiv 0$.

Part b) of the preceding theorem is already covered in Theorem 1.1 and does not depend on the symmetry of J.

The assumption that u is uniformly continuous in Theorems 1.1 and 1.3 can be weakened by requiring that u be just continuous if f satisfies, in addition to (1.5),

(1.8) for all $x \in \mathbb{R}$ the function $u \mapsto u - f(x, u)$ is not constant on any interval,

that is, for all x and all $0 \leq u_1 < u_2$ either $u_1 - f(x, u_1) \neq u_2 - f(x, u_2)$ or $u_1 - f(x, u_1) = u_2 - f(x, u_2)$ but there exists $u_3 \in (u_1, u_2)$ such that $u_3 - f(x, u_3) \neq u_1 - f(x, u_1)$. See Lemma 3.1 in Section 3.

Recently, considering a non-periodic nonlinearity f, Berestycki, Hamel and Rossi [3] analyzed the analogue of Theorem 1.3 for general elliptic operators in $\mathbb{R}^N$, finding sufficient conditions that ensure existence and uniqueness of a positive bounded solution. It is natural to ask whether the periodicity of f and the symmetry of J are crucial hypotheses in Theorem 1.3. We believe that this is the case, since a general non-local operator like (1.2) may contain a transport term and a standing wave connecting two equilibria of the system could appear. We shall investigate further this issue in a forthcoming work.

Hypothesis (1.4) implies that the operator $\mathcal{M}$ satisfies the strong maximum principle. Suppose, for instance, that J satisfies (1.3), (1.4). If $u \in C(\mathbb{R})$ satisfies $\mathcal{M}[u] \geq 0$ in $\mathbb{R}$ then u cannot achieve a global maximum without being constant. See [9]. However will need the following version:

Theorem 1.4. *Assume J satisfies (1.3), (1.4) and let $c \in L^\infty(\mathbb{R})$. If $u \in L^\infty(\mathbb{R})$ satisfies $u \leq 0$ a.e. and $\mathcal{M}[u] + c(x)u \geq 0$ a.e. in $\mathbb{R}$, then ess $sup_K u < 0$ for all compact $K \subset \mathbb{R}$ or $u = 0$ a.e. in $\mathbb{R}$.*

In Section 2 we review some spectral theory and give the argument of Theorem 1.2. Then we prove Theorem 1.1 in Section 3 and the uniqueness result, Theorem 1.3 part a) in Section 4. We leave for the appendix a proof of Theorem 1.4.

2. Some spectral Theory

In this section we deal with the principal eigenvalue problem (1.6). Before stating our result, let us recall some basic spectral results for positive operators due to Edmunds, Potter and Stuart [11] which are extensions of the Krein-Rutman theorem for positive non-compact operators.

A cone in a real Banach space X is a non-empty closed set K such that for all $x, y \in K$ and all $\alpha \geq 0$ one has $x + \alpha y \in K$, and if $x \in K$, $-x \in K$ then $x = 0$. A cone K is called reproducing if $X = K - K$. A cone K induces a partial ordering in X by the relation $x \leq y$ if and only if $x - y \in K$. A linear map or operator $T : X \to X$ is called positive if $T(K) \subseteq K$. The dual cone K^* is the set of functionals $x^* \in X^*$ which are positive, that is, such that $x^*(K) \subset [0, \infty)$.

If $T : X \to X$ is a bounded linear map on a complex Banach space X, its essential spectrum (according to Browder [5]) consists of those λ in the spectrum of T such that at least one of the following conditions holds : (1) the range of $\lambda I - T$ is not closed, (2) λ is a limit point of the spectrum of A, (3) $\cup_{n=1}^{\infty} ker((\lambda I - T)^n)$ is infinite dimensional. The radius of the essential spectrum of T, denoted by $r_e(T)$, is the largest value of $|\lambda|$ with λ in the essential spectrum of T. For more properties of $r_e(T)$ see [20].

Theorem 2.1. *(Edmunds, Potter, Stuart)*
Let K be a reproducing cone in a real Banach space X, and let $T \in \mathcal{L}(X)$ be a positive operator such that $T^p(u) \geq cu$ for some $u \in K$ with $\|u\| = 1$, some positive integer p and some positive number c. Then if $c^{\frac{1}{p}} > r_e(T_c)$, T has an eigenvector $v \in K$ with associated eigenvalue $\rho \geq c^{\frac{1}{p}}$ and T^* has eigenvector $v^* \in K^*$ corresponding to the eigenvalue ρ. Moreover ρ is unique.

A proof of this Theorem can be found in [11].

Proof of Theorem 1.2.

For convenience, in this proof we write the eigenvalue problem

$$\mathcal{M}[u] + a(x)u = -\lambda u$$

in the form

$$\mathcal{L}[u] + b(x)u = \mu u \tag{2.1}$$

where

$$\mathcal{L}[u] = J \star u, \quad b(x) = a(x) + k, \quad \mu = -\lambda + 1 + k$$

and $k > 0$ is a constant such that $\inf_{[-R,R]} b > 0$.

Observe that $\mathcal{L} : C_{per(\mathbb{R})} \to C_{per}(\mathbb{R})$ is compact ($C_{per}(\mathbb{R})$ is endowed with the norm $\|u\|_{L^\infty([-R,R])}$). Indeed, let $u_n \in C_{per}(\mathbb{R})$ be a bounded sequence, say $\|u\|_{L^\infty([-R,R])} \leq B$. Let $\epsilon > 0$ and A be large enough so that $\int_{|x|\geq A} J \leq \epsilon$. Since J is uniformly continuous in $[-R-2A, R+2A]$ there is $\delta > 0$ such that $|J(z_1) - J(z_2)| \leq \frac{\epsilon}{2(A+R)}$ for $z_1, z_2 \in [-R-2A, R+2A]$ with $|z_1 - z_2| \leq \delta$. Then for $x_1, x_2 \in [-R, R]$

$$\begin{aligned}
|\mathcal{L}[u_n](x_1) - \mathcal{L}[u_n](x_2)| &\leq \int_{\mathbb{R}} |J(x_1 - y) - J(x_2 - y)| \, |u_n(y)| \, dy \\
&\leq 2B\epsilon + B \int_{-R-A}^{R+A} |J(x_1 - y) - J(x_2 - y)| \, dy \\
&\leq 3B\epsilon.
\end{aligned}$$

Let us now establish some useful

Lemma 2.2. *Suppose $b(x) \in C^{0,1}(\mathbb{R})$ is $2R$-periodic, $b(x) > 0$ and let $\sigma := \max_{[-R,R]} b(x)$. Then $\exists p \in \mathbb{N}, \delta > 0$ and $u \in C_{per}(\mathbb{R})$, $u \geq 0$, $u \not\equiv 0$, such that*

$$\mathcal{L}^p u + b(x)^p u \geq (\sigma^p + \delta)u.$$

Observe that the proof of Theorem 1.2 will then easily follow from the above Lemma. Indeed, if the Lemma holds then since u and b are non-negative and $\mathcal{L}$ is a positive operator, we easily see that

$$(\mathcal{L} + b(x))^p[u] \geq \mathcal{L}^p[u] + b(x)^p u \geq (\sigma^p + \delta)u.$$

Using the compactness of the operator $\mathcal{L}$, we have $r_e(\mathcal{L}+b(x)) = r_e(b(x))=\sigma$, thus $(\sigma^p+\delta)^{\frac{1}{p}} > r_e(\mathcal{L}+b(x))$ and Theorem 2.1 applies.

Finally we observe that the principal eigenvalue is simple since the cone of positive $2R-$periodic functions has non-empty interior and, for a sufficiently large p, the operator $(\mathcal{L}+b)^p$ is strongly positive, that is, maps $u\geq 0$, $u\not\equiv 0$ to a strictly positive function, see [22].

□

Let now turn our attention to the proof of the above Lemma.

Proof of the Lemma:

Recall that for $p\in\mathbb{N}\setminus\{0\}$ $J\star^p u := J\star(J\star^{p-1} u)$ is well defined by induction and satisfies $J\star^p u = \mathcal{J}_p\star u$ with $\mathcal{J}_p$ defined as follows

$$\mathcal{J}_p := \underbrace{J\star J\star\ldots\star J\star J.}_{p \text{ times}}$$

By (1.4) it follows there exists $p\in\mathbb{N}$ such that $(-2R-1, 2R+1)\subset supp(\mathcal{J}_p)$.

Using the definition of $\mathcal{L}$, a short computation shows that

$$\mathcal{L}^p[u] := \int_{-R}^{R}\widetilde{\mathcal{J}}_p(x,y)u(y)\,dy,$$

with $\widetilde{\mathcal{J}}_p(x,y) = \sum_{k\in\mathbb{Z}}\mathcal{J}_p(x+2kR-y)$. Following the idea of Hutson et al. [14], consider now the following function

$$v(x) := \begin{cases} \frac{\eta(x)}{b^p(x_0)-b^p(x)+\gamma} & \text{in } \Omega_{2\epsilon} := (x_0-2\epsilon, x_0+2\epsilon)\\ 0 & \text{elsewhere,}\end{cases}$$

where $x_0\in(-R,R)$ is a point of maximum of $b(x)$, $\epsilon>0$ is chosen such that $(x_0-2\epsilon, x_0+2\epsilon)\subset(-R,R)$, γ is a positive constant that we will define later on and η is a smooth function such that $0\leq\eta\leq 1$, $\eta(x)=1$ for $|x-x_0|\leq\epsilon$, $\eta(x)=0$ for $|x-x_0|\geq 2\epsilon$. Let us compute $\mathcal{L}^p[v]+b^p(x)v-\sigma^p v$:

$$\begin{aligned}\mathcal{L}^p[v]+b^p(x)v-\sigma^p v &= \int_{x_0-\epsilon}^{x_0+\epsilon}\widetilde{\mathcal{J}}_p(x,y)\frac{dy}{b^p(x_0)-b^p(y)+\gamma} + \int_{\Omega_{2\epsilon}\setminus\Omega_\epsilon}\widetilde{\mathcal{J}}_p(x,y)\eta(y)\,dy\\ &\quad + (b^p(x)-b^p(x_0))v\\ &\geq \int_{x_0-\epsilon}^{x_0+\epsilon}\widetilde{\mathcal{J}}_p(x,y)\frac{dy}{b^p(x_0)-b^p(y)+\gamma} + (b^p(x)-b^p(x_0))v\\ &\geq \int_{x_0-\epsilon}^{x_0+\epsilon}\mathcal{J}_p(x-y)\frac{dy}{b^p(x_0)-b^p(y)+\gamma} - 1.\end{aligned}$$

Using that $(-2R-1, 2R+1)\subset supp(\mathcal{J}_p)$, it follows that $\widetilde{\mathcal{J}}_p(x,y)\geq c>0$ for $x,y\in(-R,R)$. Hence

$$\int_{x_0-\epsilon}^{x_0+\epsilon}\widetilde{\mathcal{J}}_p(x,y)\frac{dy}{b^p(x_0)-b^p(y)+\gamma}\geq c\int_{x_0-\epsilon}^{x_0+\epsilon}\frac{dy}{k|x_0-y|+\gamma},$$

where k is b Lipschitz constant for b^p. Using this inequality in the above estimate yields

$$\mathcal{L}^p[v]+b^p(x)v-\sigma^p v\geq c\int_{x_0-\epsilon}^{x_0+\epsilon}\frac{dy}{k|x_0-y|+\gamma} - 1.$$

Therefore we have

$$\begin{aligned}\mathcal{L}^p[v] + b^p(x)v - (\sigma^p + \delta)v &\geq \frac{2c}{k}\log\left(1 + \frac{k\epsilon}{\gamma}\right) - 1 - \delta v \\ &\geq \frac{2c}{k}\log\left(1 + \frac{k\epsilon}{\gamma}\right) - 1 - \frac{\delta}{\gamma}.\end{aligned}$$

Choosing now $\gamma > 0$ small so that $\frac{2c}{k}\log(1 + \frac{k\epsilon}{\gamma}) - 1 > \frac{1}{2}$ and $\delta = \frac{\gamma}{4}$, we end up with

$$\mathcal{L}^p[v] + b^p(x)v - (\sigma^p + \delta)v \geq \frac{1}{4} > 0.$$

□

3. Existence of solutions

Proof of Theorem 1.1

We follow the argument developed by Berestycki, Hamel and Roques in [2].

First assume that $\lambda_1 < 0$.

From Theorem 1.2 there exists a positive eigenfunction ϕ_1 such that

$$\mathcal{M}[\phi_1] + f_u(x,0)\phi_1 = -\lambda_1\phi_1 \geq 0.$$

Computing $\mathcal{M}[\epsilon\phi_1] + f(x, \epsilon\phi_1)$, it follows

$$\begin{aligned}\mathcal{M}[\epsilon\phi_1] + f(x,\epsilon\phi_1) &= f(x,\epsilon\phi_1) - f_u(x,0)\epsilon\phi_1 - \lambda_1\epsilon\phi_1 \\ &= -\lambda_1\epsilon\phi_1 + o(\epsilon\phi_1) > 0.\end{aligned}$$

Therefore, for $\epsilon > 0$ small, $\epsilon\phi_1$ is a periodic subsolution of (1.1). By definition of f, any constant M sufficiently large is a periodic supersolution of the problem. Choosing M so large that $\epsilon\phi_1 \leq M$ and using a basic iterative scheme yields the existence of a positive periodic solution u of (1.1).

Let now turn our attention to the non-existence setting, and assume that $\lambda_1 \geq 0$. Let u be a bounded uniformly continuous solution of (1.1). Observe that $\gamma\phi_1$ is a periodic supersolution for any positive γ. Indeed,

$$\begin{aligned}\mathcal{M}[\gamma\phi_1] + f(x,\gamma\phi_1) &< \mathcal{M}[\gamma\phi_1] + f_u(x,0)\gamma\phi_1 \\ &< -\lambda_1\gamma\phi_1 \leq 0.\end{aligned}$$

Since $\phi_1 \geq \delta$ for some positive δ we may define the following quantity

$$\gamma^* := \inf\{\gamma > 0 | u \leq \gamma\phi_1\}.$$

We claim that

Claim 3.1. $\gamma^* = 0$

Observe that we end the proof of the theorem by proving the above claim.

Proof of the Claim:

Assume that $\gamma^* > 0$, then by definition of γ^* we have the following possibilities:

- Either there exists $x_0 \in \mathbb{R}$ such that $\gamma^*\phi_1(x_0) = u(x_0)$
- Or there exists a sequence of point $(x_n)_{n\in\mathbb{N}}$ such that $|x_n| \to +\infty$ and $\lim_{n\to+\infty} \gamma^*\phi_1(x_n) - u(x_n) = 0$.

In the first case we have,

$$0 \le \mathcal{M}[w](x_0) = \mathcal{M}[(\gamma^*\phi_1 - u)](x_0) + f(x_0, \gamma^*\phi_1(x_0)) - f(x_0, u(x_0)) \le 0.$$

By the maximum principle we end up with $\gamma^*\phi_1 \equiv u$ and get the following contradiction,

$$0 = \mathcal{M}[\gamma^*\phi_1] + f(x, \gamma^*\phi_1) < \mathcal{M}[\gamma^*\phi_1] + f_u(x, 0)\gamma^*\phi_1 \le 0.$$

Hence $\gamma^* = 0$.

In the second case we argue as follows. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence of points satisfying for all $n, y_n \in (-R, R)$ and $x_n - y_n \in R\mathbb{Z}$. Up to extraction of a subsequence, $y_n \to \bar{y}$. Now consider the following sequence of functions $u_n := u(. + x_n)$, $\phi_n := \phi_1(. + x_n)$ and $w_n := \gamma^*\phi_n - u_n$. Since $\mathcal{M}$ is translation invariant and f periodic, u_n and $\phi_n > 0$ satisfy

$$\mathcal{M}[u_n] + f(x + y_n, u_n) = 0$$
$$\mathcal{M}[\gamma^*\phi_n] + f(x + y_n, \gamma^*\phi_n) > 0.$$

Since, u and ϕ_1 are uniformly continuous, extracting a subsequence we may assume that $(u_n)_{n\in\mathbb{N}}$ and $(\phi_n)_{n\in\mathbb{N}}$ converge uniformly on compact sets to some functions $\bar{u}$ and $\bar{\phi}$ which satisfy

$$\mathcal{M}[\bar{u}] + f(x + \bar{y}, \bar{u}) = 0$$
$$\mathcal{M}[\gamma^*\bar{\phi}] + f(x + \bar{y}, \gamma^*\bar{\phi}) > 0.$$

Moreover, $w_n \to \bar{w} = \gamma^*\bar{\phi} - \bar{u}$ uniformly on compact sets. Note that $\bar{w} \ge 0$ and achieves a global minimum at 0, since $\bar{w}(0) = \lim_{n\to\infty}(\gamma^*\phi_n - u_n)(0) = \lim_{n\to\infty}(\gamma^*\phi_1 - u)(x_n) = 0$. Arguing now as above, we get the desired contradiction. Hence, $\gamma^* = 0$.

□

As mentioned earlier, the hypothesis that u is uniformly continuous can be relaxed under an additional condition on f.

Lemma 3.1. *Assume J satisfies* (1.3) *and f is continuous and satisfies* (1.8). *Then any bounded continuous solution to* (1.1) *is uniformly continuous.*

Proof.

Since u is bounded and J is $C(\mathbb{R}) \cap L^1(\mathbb{R})$ the function $J \star u$ is uniformly continuous on $\mathbb{R}$. Suppose u is not uniformly continuous. Then there is $\delta > 0$ and sequences x_n, y_n, which we may assume converge to $+\infty$, such that $|x_n - y_n| \to 0$ and

$$u(y_n) - u(x_n) \ge \delta.$$

Working with a subsequence we may also assume that $u(x_n) \to u_1$, $u(y_n) \to u_2$ and hence $u_2 - u_1 \ge \delta$. Since $J \star u$ is uniformly continuous we have $J \star u(x_n) - J \star u(y_n) \to 0$. Write $g(x, u) = u - f(x, u)$ so that equation (1.1) is equivalent to

$$J \star u = g(x, u).$$

It follows that

$$g(x_n, u(x_n)) - g(y_n, u(y_n)) \to 0 \quad \text{as } n \to +\infty.$$

Using that g is periodic in x we have $g(x_n, \cdot) = g(\widetilde{x}_n, \cdot)$, $g(y_n, \cdot) = g(\widetilde{y}_n, \cdot)$ where $\widetilde{x}_n, \widetilde{y}_n$ are bounded. Hence, considering a new subsequence we can assume $\lim \widetilde{x}_n = \lim \widetilde{y}_n = x_0$. By hypothesis $g(x_0, \cdot)$ is not constant on the interval $[u_1, u_2]$ and hence there exists $u_3 \in (u_1, u_2)$ such that $g(x_0, u_3) \neq g(x_0, u_1)$. By continuity, for sufficiently large n, there is z_n with $|z_n - x_n| \to 0$ such that $u(z_n) = u_3$. But then, since $J \star u$ is uniformly continuous

$$g(z_n, u_3) - g(x_n, u(x_n)) = J \star u(z_n) - J \star u(x_n) \to 0. \tag{3.1}$$

Letting $\widetilde{z}_n$ be such that $\widetilde{z}_n \to x_0$ and $g(z_n, \cdot) = g(\widetilde{z}_n, \cdot)$ we reach that

$$\lim g(z_n, u_3) - g(x_n, u(x_n)) = g(x_0, u_3) - g(x_0, u_1) \neq 0,$$

which contradicts (3.1). □

4. Uniqueness when J is symmetric

Throughout this section we assume that J is symmetric. For the proof of Theorem 1.3 we follow the ideas in [2].

Proof of Theorem 1.3

Part b) of this theorem is contained in Theorem 1.1 so we concentrate on part a).

Let p denote the positive periodic solution to (1.1) constructed in Theorem 1.1 and let $u \geq 0$, $u \not\equiv 0$ be a bounded, continuous solution. We will prove that $u \equiv p$.

We show first that $u \leq p$. Set

$$\gamma^* := \inf\{\gamma > 0 | u \leq \gamma p\}.$$

Note that γ^* is well defined because u is bounded and p is bounded below by a positive constant. We claim that

$$\gamma^* \leq 1.$$

Suppose that $\gamma^* > 1$ and note that $u \leq \gamma^* p$. If there exists $x_0 \in \mathbb{R}$ such that $\gamma^* p(x_0) = u(x_0)$ then by the strong maximum principle $\gamma^* p \equiv u$ and we obtain

$$0 = \mathcal{M}[\gamma^* p] + f(x, \gamma^* p).$$

Hence $f(x, \gamma^* p) = \gamma^* f(x, p)$ for all $x \in \mathbb{R}$ and this implies $\gamma^* = 1$.

If there is no x_0 as above then

$$\gamma^* p - u > 0 \quad \text{ in } \mathbb{R}$$

and there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $|x_n| \to +\infty$ and $\lim_{n\to+\infty} \gamma^* p(x_n) - u(x_n) = 0$. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence satisfying $y_n \in [-R, R]$ and $x_n - y_n = k_n 2R$ for some $k_n \in \mathbb{Z}$. We may assume that $y_n \to \bar{y}$. Let $u_n := u(. + x_n)$ which satisfies

$$\mathcal{M}[u_n] + f(x + y_n, u_n) = 0.$$

Let $w_n = \gamma^* p(. + y_n) - u_n \geq 0$. Then $w_n > 0$ in $\mathbb{R}$ and

$$J \star w_n = a_n(x) w_n$$

where

$$a_n(x) = 1 - \frac{\gamma^* f(x + y_n, p(x + y_n)) - f(x + y_n, u_n(x))}{\gamma^* p(x + y_n) - u_n(x)}.$$

Since $w_n > 0$ we deduce a_n is well defined and $a_n \geq 0$. Using that $f(x, u)/u$ is nonincreasing with respect to u and the fact that $\gamma^* > 1$ we have $f(x, \gamma^* p) \leq \gamma^* f(x, p)$. This implies

$$\frac{\gamma^* f(x, p) - f(x, u)}{\gamma^* p - u} \geq \frac{f(x, \gamma^* p) - f(x, u)}{\gamma^* p - u} \geq -C.$$

Thus

$$0 \le a_n \le C + 1 \quad \text{in } \mathbb{R}, \text{ for all } n,$$

with C independent of n. Observe that

$$J \star w_n(0) = a_n(0)(\gamma^* p(y_n) - u(x_n)) = a_n(0)(\gamma^* p(x_n) - u(x_n)) \to 0$$

which implies

$$\int_{\mathbb{R}} J(-y) w_n(y)\, dy \to 0 \quad \text{as } n \to +\infty.$$

Similarly,

$$J \star J \star w_n(0) = J \star (a_n w_n)(0) = \int_{\mathbb{R}} J(-y) a_n(y) w_n(y)\, dy$$

but

$$\left| \int_{\mathbb{R}} J(-y) a_n(y) w_n(y)\, dy \right| \le \|a_n\|_{L^\infty} \int_{\mathbb{R}} J(-y) w_n(y)\, dy \to 0.$$

Hence

$$J \star J \star w_n(0) = \int_{\mathbb{R}} (J \star J)(-y) w_n(y)\, dy \to 0 \quad \text{as } n \to +\infty.$$

Defining

$$\mathcal{J}_k := \underbrace{J \star \ldots \star J}_{k \text{ times}}$$

we see that for all $k \in \mathbb{N}$

$$\int_{\mathbb{R}} \mathcal{J}_k(-y) w_n(y)\, dy \to 0 \quad \text{as } n \to +\infty.$$

Hypothesis (1.4) implies that the support of $\mathcal{J}_k$ converges to all of $\mathbb{R}$ as $k \to +\infty$. Therefore, for a subsequence, $w_n \to 0$ a.e. in $\mathbb{R}$ as $n \to +\infty$. Since p is periodic and continuous $p(x + y_n) \to p(x + \bar{y})$ uniformly with respect to x. Hence, $\bar{u}(x) = \lim_{n\to+\infty} u_n(x)$ exists a.e. and by dominated convergence $\bar{u}$ is a solution to

$$\mathcal{M}[\bar{u}] + f(x + \bar{y}, \bar{u}) = 0. \tag{4.1}$$

But since $w_n \to 0$ a.e. we have $\bar{u} = \gamma^* p(\cdot + \bar{y})$. Thus $\gamma^* p(\cdot + \bar{y})$ is a solution to (4.1) which is impossible for $\gamma^* > 1$ as argued before.

The proof that $p \le u$ is analogous, but a key point is to prove first that under the conditions of Theorem 1.3 any nontrivial, nonnegative solution is bounded below by a positive constant. This is the content of Proposition 4.1 below.

□

Proposition 4.1. Assume that J satisfies (1.3), (1.4) and (1.7), f satisfies (1.5) and that the operator $-(\mathcal{M} - f_u(x, 0))$ has a negative principal periodic eigenvalue. Suppose that u is a non-negative, bounded, continuous solution to (1.1). Then $u \equiv 0$ or there exists a constant $c > 0$ such that

$$u(x) \ge c \quad \text{for all } x \in \mathbb{R}.$$

The basic tool to prove Proposition 4.1, following an idea in [2], is to study the principal eigenvalue of the linearized operator in bounded domains. More precisely let $\Omega = (-r, +r)$ and $a\colon \Omega \to \mathbb{R}$ be Lipschitz. We consider the eigenvalue problem in Ω with "Dirichlet boundary condition" in the following sense:

$$
\begin{cases} \mathcal{M}[\varphi] + a(x)\varphi = -\lambda\varphi & \text{in } \Omega \\ \varphi(x) = 0 \quad \text{for all } x \notin \Omega \\ \varphi \in C(\overline{\Omega}). \end{cases} \tag{4.2}
$$

We show that the principal eigenvalue for (4.2) exists and converges to the principal periodic eigenvalue as $r \to +\infty$. The first step is to establish variational characterizations of these eigenvalues, which is the argument that requires the symmetry of J.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}$ be a bounded open interval. Assume that J satisfies* (1.3), (1.4) *and* (1.7) *and let $a\colon \Omega \to \mathbb{R}$ be Lipschitz. Then there exists a smallest λ_1 such that* (4.2) *has a nontrivial solution. This eigenvalue is simple and the eigenfunctions are of constant sign in Ω. Moreover*

$$
\lambda_1 = \min_{\varphi \in C(\overline{\Omega})} -\frac{\int_\Omega (\mathcal{M}[\widetilde{\varphi}] + a(x)\varphi)\varphi}{\int_\Omega \varphi^2}, \tag{4.3}
$$

where $\widetilde{\varphi}$ denotes the extension by 0 of φ to $\mathbb{R}$, and the minimum is attained.

The statement and proof are analogous to that of Theorem 3.1 in [14] except that here we do not assume that $J(0) > 0$. A different formula for the principal eigenvalue with Dirichlet boundary condition appears in [7], where it is used to characterize the rate of decay of solutions to a linear evolution equation.

Proof.

Define the operator $X[\varphi] = \int_\Omega J(x-y)\,\varphi(y)\,dy$ for $\varphi \in C(\overline{\Omega})$. Then $X : C(\overline{\Omega}) \to C(\overline{\Omega})$ is compact. Let $c_0 > 0$ be such that $\inf_\Omega a(x) + c_0 > 0$ and define $\widetilde{a} = a + c_0$. The eigenvalue problem (4.2) is equivalent to: find $\varphi \in C(\overline{\Omega})$ and $\lambda \in \mathbb{R}$ such that

$$
X[\varphi] + \widetilde{a}\varphi = (-\lambda + 1 + c_0)\varphi \quad \text{in } \Omega.
$$

A calculation similar to Lemma 2.2 shows that there exists an integer p, $u \in C(\overline{\Omega})$ and $\delta > 0$ such that

$$
(X + \widetilde{a})^p u \geq ((\max_{\overline{\Omega}} \widetilde{a})^p + \delta)u \quad \text{in } \Omega. \tag{4.4}
$$

Using Theorem 2.1 we deduce that the operator $X + \widetilde{a}$ has a unique principal eigenvalue $\rho > 0$ and a principal eigenvector $\varphi_1 \in C(\overline{\Omega})$. Let $\lambda = 1 + c_0 - \rho$ so that $X[\varphi_1] + a(x)\varphi_1 = (1-\lambda)\varphi_1$. From (4.4) we deduce that σ_+ defined by

$$
\sigma_+ = \sup_{\varphi \in C(\overline{\Omega})} \frac{\int_\Omega (X[\varphi] + a(x)\varphi)\varphi}{\int_\Omega \varphi^2} \tag{4.5}
$$

satisfies

$$
\sigma_+ \geq 1 - \lambda > \max_{\overline{\Omega}} a. \tag{4.6}
$$

Now, using the same argument as in [14] we deduce that supremum in (4.5) is achieved. Indeed, it is standard [4] that the spectrum of $X + a(x)$ is to the left of σ_+ and that there exists a sequence $\varphi_n \in C(\overline{\Omega})$ such that $\|\varphi\|_{L^2(\Omega)} = 1$ and $\|(X + a(x) - \sigma_+)\varphi_n\|_{L^2(\Omega)} \to 0$ as $n \to +\infty$. By compactness of $X : L^2(\Omega) \to C(\overline{\Omega})$ for a subsequence $\lim_{n\to+\infty} X[\varphi_n]$ exists in $C(\overline{\Omega})$. Then, using (4.6), we see that

$\varphi_n \to \varphi$ in $C(\overline{\Omega})$ for some φ and $(X+a)\varphi = \sigma_+\varphi$. Thus σ_+ is a principal eigenvalue for the operator X and by uniqueness of this eigenvalue we have $\sigma_+ = 1-\lambda$. $\square$

Lemma 4.3. *Assume J satisfies* (1.3), (1.4) *and* (1.7) *and that $a\colon \mathbb{R}\to\mathbb{R}$ is a $2R$-periodic, Lipschitz function. Then the principal eigenvalue of the operator $-(\mathcal{M}+a(x))$ in $C_{per}(\mathbb{R})$ is given by*

$$\lambda_1(a) = \inf_{\|\varphi\|_{L^2(\mathbb{R})}=1} -\int_{\mathbb{R}} (\mathcal{M}[\varphi]+a(x)\varphi)\varphi \tag{4.7}$$

$$= \min_{\varphi\in C_{per}(\mathbb{R})} -\frac{\int_{-R}^{R}(\mathcal{M}[\varphi]+a(x)\varphi)\varphi}{\int_{-R}^{R}\varphi^2}. \tag{4.8}$$

Proof.

By Theorem 1.2 we know that there exists a unique principal eigenvalue $\lambda_1(a)$ of the operator $-(\mathcal{M}+a)$ in $C_{per}(\mathbb{R})$. Let $\phi_1\in C_{per}(\mathbb{R})$ denote a positive eigenfunction associated with $\lambda_1(a)$. We normalize ϕ_1 such that

$$\int_{-R}^{R}\phi_1^2 = 2R. \tag{4.9}$$

On the other hand the quantity

$$\widetilde{\lambda}_1(a) = \inf_{\varphi\in C_{per}(\mathbb{R})} -\frac{\int_{-R}^{R}(\mathcal{M}[\varphi]+a(x)\varphi)\varphi}{\int_{-R}^{R}\varphi^2}$$

is also an eigenvalue of $-(\mathcal{M}+a)$ on $C_{per}(\mathbb{R})$ with a positive eigenfunction. By uniqueness of the principal eigenvalue $\lambda_1(a)=\widetilde{\lambda}_1(a)$.

We claim:

$$\inf_{\|\varphi\|_{L^2(\mathbb{R})}=1} -\int_{\mathbb{R}}(\mathcal{M}[\varphi]+a(x)\varphi)\varphi \le \lambda_1(a).$$

Indeed, for $r>0$ let $\eta_r\in C_0^\infty(\mathbb{R})$ be such that $0\le\eta_r\le 1$, $\eta_r(x)=1$ for $|x|\le r$, $\eta_r(x)=0$ for $|x|\ge r+1$. It will be sufficient to show that

$$\lim_{r\to+\infty}\frac{\int_{\mathbb{R}}(\mathcal{M}[\phi_1\eta_r]+a\phi_1\eta_r)\phi_1\eta_r}{\int_{\mathbb{R}}(\phi_1\eta_r)^2} = -\lambda_1(a). \tag{4.10}$$

By (4.9) we have

$$\int_{\mathbb{R}}(\phi_1\eta_r)^2 = 2r+O(1) \qquad \text{as } r\to+\infty. \tag{4.11}$$

Let $0<\theta<1$. Then

$$\begin{aligned}
|\mathcal{M}[\phi_1](x)-\mathcal{M}[\phi_1\eta_r]| &\le \|\phi_1\|_{L^\infty}\int_{|x-z|\ge r}|J(z)|\,dz\\
&\le \|\phi_1\|_{L^\infty}\int_{|z|\ge(1-\theta)r}|J(z)|\,dz \quad \text{for all } |x|\le\theta r\\
&= o(1) \quad \text{uniformly for all } |x|\le\theta r.
\end{aligned} \tag{4.12}$$

We split the integral

$$\int_{\mathbb{R}}(\mathcal{M}[\phi_1\eta_r]+a\phi_1\eta_r)\phi_1\eta_r = \int_{|x|\le\theta r}\dots dx + \int_{|x|\ge\theta r}\dots dx. \tag{4.13}$$

Using $\eta_r(x) = 1$ for $|x| \le \theta r$ and (4.12) we see that

$$\begin{aligned}\int_{|x|\le\theta r} (\mathcal{M}[\phi_1\eta_r] + a\phi_1\eta_r)\phi_1\eta_r &= \int_{|x|\le\theta r} (\mathcal{M}[\phi_1\eta_r] + a\phi_1)\phi_1 \\ &= \int_{|x|\le\theta r} (\mathcal{M}[\phi_1] + a\phi_1 + o(1))\phi_1 \\ &= -2\theta\lambda_1(a)r + o(r) \quad \text{ as } r \to +\infty.\end{aligned}$$

The second integral in (4.13) is bounded by

$$\left| \int_{|x|\ge\theta r} (\mathcal{M}[\phi_1\eta_r] + a\phi_1\eta_r)\phi_1\eta_r \right| \le C(1-\theta)r. \tag{4.14}$$

Thus from (4.11)–(4.14) we conclude that

$$\left| \frac{\int_{\mathbb{R}} (\mathcal{M}[\phi_1\eta_r] + a\phi_1\eta_r)\phi_1\eta_r}{\int_{\mathbb{R}} (\phi_1\eta_r)^2} + \lambda_1(a) \right| \le C(1-\theta) + o(1)$$

which proves (4.10).

To establish (4.7) it rests to verify that

$$\lambda_1(a) \le -\frac{\int_{\mathbb{R}} (\mathcal{M}[\varphi] + a(x)\varphi)\varphi}{\int_{\mathbb{R}} \varphi^2} \quad \text{ for all } \varphi \in C_c(\mathbb{R}). \tag{4.15}$$

By uniqueness of the principal eigenvalue we have

$$\lambda_1(a) = \inf_{\varphi \in C_{per}(\Omega_k)} -\frac{\int_{-kR}^{kR} (\mathcal{M}[\varphi] + a(x)\varphi)\varphi}{\int_{-kR}^{kR} \varphi^2}. \tag{4.16}$$

where

$$\Omega_k = (-kR, kR) \quad \text{ for } k \ge 1$$

and $C_{per}(\Omega_k)$ is the set of continuous $2kR$-periodic functions on $\mathbb{R}$.

Fix $\varphi \in C_c(\mathbb{R})$ and consider k large enough so that $supp(\varphi) \subseteq \Omega_k$. Consider now φ_k the $4kR$-periodic extension of φ. Since $\varphi_k \in C_{per}(\Omega_{2k})$ (4.16) yields

$$\lambda_1(a) \le -\frac{\int_{-2kR}^{2kR} (\mathcal{M}[\varphi_k] + a(x)\varphi_k)\varphi_k}{\int_{-2kR}^{2kR} \varphi_k^2} = -\frac{\int_{\mathbb{R}} (\mathcal{M}[\varphi_k] + a(x)\varphi)\varphi}{\int_{\mathbb{R}} \varphi^2}. \tag{4.17}$$

For $|x| \le kR$ we have

$$|\mathcal{M}[\varphi_k](x) - \mathcal{M}[\varphi](x)| \le \|\varphi\|_{L^\infty} \int_{|y|\ge 2kR} |J(x-y)|\, dy \le \|\varphi\|_{L^\infty} \int_{|z|\ge kR} |J(z)|\, dz$$

Hence

$$\lim_{k\to+\infty} \int_{\mathbb{R}} (\mathcal{M}[\varphi_k] + a(x)\varphi)\varphi = \int_{\mathbb{R}} (\mathcal{M}[\varphi] + a(x)\varphi)\varphi. \tag{4.18}$$

Thanks to (4.17) and (4.18) we conclude the validity of (4.15). □

Lemma 4.4. *Assume J satisfies* (1.3), (1.4) *and* (1.7) *and that $a\colon \mathbb{R} \to \mathbb{R}$ is a $2R$-periodic, Lipschitz function. Let $\lambda_{r,y}$ be the principal eigenvalue of* (4.2) *for*

$$\Omega_{r,y} = B_r(y)$$

and let $\lambda_1(a)$ denote the principal eigenvalue of $-(\mathcal{M} + a(x))$ in $C_{per}(\mathbb{R})$. Then

$$\lim_{r\to+\infty} \lambda_{r,y} = \lambda_1(a).$$

Moreover, the applications $y \mapsto \lambda_{r,y}$ and $y \mapsto \varphi_{r,y}$ are periodic. The periodicity of the application $y \mapsto \varphi_{r,y}$ is understood as follows

$$\varphi_{r,y+2R}(x) = \varphi_{r,y}(x - 2R).$$

Proof.

For convenience we write

$$\lambda_r = \lambda_{r,y}$$

and let φ_r be a positive eigenfunction of (4.2) in Ω_r.

By the variational characterization (4.3) we see that $r \mapsto \lambda_r$ is non-increasing and hence $\lim_{r\to+\infty} \lambda_r$ exists. Moreover, using (4.7) we have

$$\lambda_r \geq \lambda_1(a) \quad \text{for all } r > 0. \tag{4.19}$$

Let $\phi_1 \in C_{per}(\mathbb{R})$ be a positive eigenfunction of $-(\mathcal{M} + a(x))$ with eigenvalue $\lambda_1(a)$ normalized such that

$$\int_{-R}^{R} \phi_1^2 = 2R.$$

Let $\eta_r \in C_0^\infty(\mathbb{R})$ be such that $0 \leq \eta \leq 1$,

$$\eta_r(x) = 1 \text{ for } |x - y| \leq r - 1, \qquad \eta_r(x) = 0 \text{ for } |x - y| \geq r$$

and such that $\|\eta_r\|_{C^2(\mathbb{R})} \leq C$ with C independent of r. Arguing in the same way as in the proof of Lemma 4.3 we obtain

$$\lim_{r\to+\infty} \frac{\int_{\mathbb{R}} (\mathcal{M}[\phi_1\eta_r] + a\phi_1\eta_r)\phi_1\eta_r}{\int_{\mathbb{R}} (\phi_1\eta_r)^2} = -\lambda_1(a).$$

Since

$$\lambda_r \leq -\frac{\int_{\mathbb{R}} (\mathcal{M}[\phi_1\eta_r] + a\phi_1\eta_r)\phi_1\eta_r}{\int_{\mathbb{R}} (\phi_1\eta_r)^2}$$

we conclude

$$\lim_{r\to+\infty} \lambda_r \leq \lambda_1(a).$$

This and (4.19) prove the desired result.

Let us now show the periodicity of the applications $y \mapsto \lambda_{r,y}$ and $y \mapsto \varphi_{r,y}$. Replace y by $y+2R$ in the above problem (4.2) and let us denote $\lambda_{r,y+2R}$ and $\varphi_{r,y+2R}$ corresponding principal eigenvalue and the associated positive eigenfunction.

$$\mathcal{M}[\varphi_{r,y+2R}] + a(x)\varphi_{r,y+2R} = -\lambda_{r,y+2R}\varphi_{r,y+2R} \qquad \text{in } B_r(y + 2R)$$

We take the following normalization

$$\int_{\Omega_{r,y+2R}} \varphi^2_{r,y+2R}(x)\, dx = 1.$$

Let us defined $\psi(x) := \varphi_{r,y+2R}(x + 2R)$ for any $x \in B_r(y)$. A short computation shows that

$$\mathcal{M}[\psi](x) = \mathcal{M}[\varphi]_{r,y+2R}(x + 2R)$$

Therefore, using the periodicity of $a(x)$, we have

$$\mathcal{M}[\psi](x) + a(x + 2R)\psi(x) = \lambda_{r,y+2R}\psi \text{ in } B_r(y)$$
$$\mathcal{M}[\psi](x) + a(x)\psi(x) = \lambda_{r,y+2R}\psi \text{ in } B_r(y)$$

Thus, $\lambda_{r,y+2R}$ is a principal eigenvalue of the problem (4.2) with $\Omega_{r,y} = B_r(y)$. Hence, by uniqueness of the principal eigenvalue we have $\lambda_{r,y} = \lambda_{r,y+2R}$ and $\psi = \gamma\varphi_{r,y}$ for some positive γ. Using the normalization, it follows that $\gamma = 1$. Therefore, $\varphi_{r,y}(x) = \varphi_{r,y+2R}(x + 2R)$, in other words

$$\varphi_{r,y+2R}(x) = \varphi_{r,y}(x - 2R).$$

$\square$

Remark 4.5. The proof of Lemma 4.4 yields the slightly stronger conclusion that the convergence

$$\lim_{r\to+\infty} \lambda_{r,y} = \lambda_1(a)$$

is uniform with respect to $y \in \mathbb{R}$.

Proof of Proposition 4.1

Let $u \geq 0$ be a bounded, continuous solution to (1.1) such that $u \not\equiv 0$. By the strong maximum principle we must have $u > 0$ in $\mathbb{R}$.

Given $y \in \mathbb{R}$ and $r > 0$ we write $\Omega_{r,y} = (y-r, y+r)$, $\lambda_{r,y}$ the principal eigenvalue of $-(\mathcal{M} + f_u(x,0))$ with Dirichlet boundary condition in $\Omega_{r,y}$ as in (4.2), and $\varphi_{r,y}$ a positive Dirichlet eigenfunction normalized so that

$$\int_{\Omega_{r,y}} \varphi_{r,y}^2 = 1.$$

Since the principal eigenvalue $\lambda_1 := \lambda_1(f_u(x,0))$ of $-(\mathcal{M} + f_u(x,0))$ with periodic boundary conditions is negative by hypothesis, by Lemma 4.4 and Remark 4.5 we may fix $r > 0$ large enough so that

$$\lambda_{r,y} < \lambda_1/2 \quad \forall y \in \mathbb{R}.$$

Note that for $x \in \Omega_{r,y}$

$$\begin{aligned}\mathcal{M}[\gamma\varphi_{r,y}] + f(x, \gamma_{r,y}\varphi_{r,y}) &= -\lambda_{r,y}\gamma\varphi_{r,y} - f_u(x,0)\gamma\varphi_{r,y} + f(x, \gamma\varphi_{r,y}) \\ &\geq -\frac{\lambda_1}{2}\gamma\varphi_{r,y} - f_u(x,0)\gamma\varphi_{r,y} + f(x, \gamma\varphi_{r,y}) \\ &\geq 0\end{aligned}$$

if $0 \leq \gamma \leq \gamma_0$ with γ_0 fixed suitably small. For $x \notin \Omega_{y,r}$ we have $\varphi_{y,r}(x) = 0$ and $\mathcal{M}[\varphi_{r,y}] \geq 0$. Thus

$$\mathcal{M}[\gamma\varphi_{r,y}] + f(x, \gamma\varphi_{r,y}) \geq 0 \quad \text{in } \mathbb{R} \tag{4.20}$$

for all $0 < \gamma < \gamma_0$.

We claim that

$$\gamma_0\varphi_{r,y} \leq u \quad \text{in } \mathbb{R}, \text{ for all } y \in \mathbb{R}. \tag{4.21}$$

This proves the proposition because there is a positive constant c such that $\varphi_{r,y}(x) \geq c$ for all $y \in \mathbb{R}$ since the application $y \mapsto \varphi_{r,y}$ is periodic and $\varphi_{r,y} > 0$ for any $y \in [-2R, 2R]$.

Now, to prove (4.21) fix $y \in \mathbb{R}$ and set

$$\gamma^* = \sup\{\, \gamma > 0 \,/\, \gamma\varphi_{r,y} \leq u \text{ in } \mathbb{R}\}.$$

Since $u > 0$ in $\mathbb{R}$ and $\varphi_{r,y}$ has compact support we see that $\gamma^* > 0$. Assume that $\gamma^* < \gamma_0$. Then $\gamma^* \varphi_{r,y} \le u$ and there must be some $\hat{x} \in \Omega_{r,y}$ such that

$$\gamma^* \varphi_{r,y}(\hat{x}) = u(\hat{x}) \tag{4.22}$$

($\varphi_{r,y}$ is continuous in $\overline{\Omega}$ so that the extension by zero outside $\overline{\Omega}$ is upper semi-continuous). Then by (4.20) $\gamma^* \varphi_{r,y}$ is a subsolution of (1.1) while u is a solution. By the strong maximum principle, Theorem 1.4, $\gamma^* \varphi_{r,y} < u$ which contradicts (4.22). □

Appendix

In this appendix we give a short proof of Theorem 1.4. We assume that J satisfies (1.3), (1.4), $c \in L^\infty(\mathbb{R})$ and $u \in L^\infty(\mathbb{R})$ satisfies

$$\begin{aligned} u &\le 0 \quad \text{a.e. in } \mathbb{R}, \\ \mathcal{M}[u] + cu &\ge 0 \quad \text{a.e. in } \mathbb{R}. \end{aligned} \tag{4.23}$$

For $\epsilon > 0$ define

$$u_\epsilon(x) = \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} u.$$

Then u_ϵ is continuous in $\mathbb{R}$, $u_\epsilon \le 0$ and $u_\epsilon \to u$ a.e. as $\epsilon \to 0$. There are 2 cases:

1) for any closed interval I one has $\limsup_{\epsilon\to 0} \sup_I u_\epsilon < 0$, or
2) for some closed interval I one has $\limsup_{\epsilon\to 0} \sup_I u_\epsilon = 0$.

If case 1) occurs, we see that for all closed intervals I we have $ess\ sup_I u < 0$. Assume case 2) holds. Let I be a closed interval and $\epsilon_n \to 0$ be such that $\lim_{n\to+\infty} u_{\epsilon_n}(x_n) = 0$. Integrating (4.23) from $x_n - \epsilon_n$ to $x_n + \epsilon_n$ an dividing by $2\epsilon_n$ we have

$$J \star u_{\epsilon_n}(x_n) \ge u_{\epsilon_n}(x_n) - \frac{1}{2\epsilon_n} \int_{x_n-\epsilon_n}^{x_n+\epsilon_n} cu.$$

But, since $u \le 0$ a.e.

$$\left| \frac{1}{2\epsilon_n} \int_{x_n-\epsilon_n}^{x_n+\epsilon_n} cu \right| \le -\|c\|_{L^\infty} u_{\epsilon_n}(x_n) \to 0.$$

Hence

$$\liminf_{n\to+\infty} J \star u_{\epsilon_n}(x_n) \ge 0.$$

We may assume that $x_n \to x \in I$. Then by dominated convergence

$$J \star u_{\epsilon_n}(x_n) = \int_{\mathbb{R}} J(x_n - y) u_{\epsilon_n}(y)\, dy \to \int_R J(x-y) u(y)\, dy.$$

This shows that $u = 0$ a.e. in $x - supp(J)$. Now, for any x_1 in the interior of $x - supp(J)$ we have $J \star u(x_1) \ge 0$, which shows that $u = 0$ a.e. in $x - 2supp(J)$. Repeating this argument and using (1.4) — which implies that $k\, supp(J)$ covers all of $\mathbb{R}$ as $k \to +\infty$ — we deduce that $u = 0$ a.e. in $\mathbb{R}$. □

Acknowledgements. This work has been partly supported by the Ecos-Conicyt project C05E04, Fondecyt grants 1050754 and 1050725, Nucleus Millenium P04-069-F Information and Randomness and by FONDAP grant for Applied Mathematics, Chile.

References

[1] P.W. Bates, P.C. Fife, X. Ren, X. Wang, *Traveling waves in a convolution model for phase transitions*, Arch. Rational Mech. Anal. 138 (1997), no. 2, 105–136.

[2] H. Berestycki, F. Hamel, L. Roques, *Analysis of the periodically fragmented environment model. I. Species persistence.* J. Math. Biol. 51 (2005), no. 1, 75–113.

[3] H. Berestycki, F. Hamel, L. Rossi, *Liouville-type results for semilinear elliptic equations in unbounded domains.* Ann. Mat. Pura Appl. (electronic)

[4] H. Brezis, Analyse fonctionnelle. Théorie et applications. Masson, Paris, 1983.

[5] F.E. Browder, *On the spectral theory of elliptic differential operators. I.* Math. Ann. 142 (1960/1961), 22–130.

[6] J. Carr, A. Chmaj, *Uniqueness of travelling waves for nonlocal monostable equations.* Proc. Amer. Math. Soc. 132 (2004), no. 8, 2433–2439.

[7] E. Chasseigne, M. Chaves, J.D. Rossi, *Asymptotic behavior for nonlocal diffusion equations.* Preprint 2006.

[8] A. Chmaj, X .Ren, *Homoclinic solutions of an integral equation: existence and stability*, J. Differential Equations 155 (1999), no. 1, 17–43.

[9] J. Coville, *Maximum principles, sliding techniques and applications to nonlocal equations.* Preprint 2006.

[10] J. Coville, L. Dupaigne, *Propagation speed of travelling fronts in non local reaction-diffusion equations*, Nonlinear Anal. 60 (2005), no. 5, 797–819.

[11] D.E. Edmunds, A.J.B. Potter, C.A. Stuart, *Non-compact positive operators.* Proc. Roy. Soc. London Ser. A 328 (1972), no. 1572, 67–81.

[12] P. Fife, Mathematical aspects of reacting and diffusing systems, Lecture Notes in Biomathematics, 28. Springer-Verlag, Berlin-New York, (1979).

[13] R.A. Fisher, *The genetical theory of natural selection.* complete variorum edition. Revised reprint of the 1930 original. Edited, with a foreword and notes, by J. H. Bennett. Oxford University Press, Oxford, (1999).

[14] V. Hutson, S. Martínez, K. Mischaikow, G.T. Vickers, *The evolution of dispersal.* J. Math. Biol. 47 (2003), no. 6, 483–517.

[15] Y.I. Kanel, *Certain problems on equations in the theory of burning*, Dokl. Akad. Nauk SSSR 136 277–280. Translated as Soviet Math. Dokl. 2 1961 48–51.

[16] A.N. Kolmogorov, I.G. Petrovsky, N.S. Piskunov, *Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, Bulletin Université d'État à Moscow (Bjul. Moskowskogo Gos. Univ), Série Internationale, Section A. 1, 1937, pp. 1–26. English Translation: Study of the Difussion Equation with Growth of the Quantity of Matter and its Application to a Biological Problem. In Dynamics of curved front, R. PELCÉ Ed., Perspective in Physics Series, Academic Press, New York, (1988), 105–130.

[17] M. Kot, M. Lewis, P. Van den Driessche, *Dispersal data and the spread of invading organisms*, Ecology 77 (7), 1996, 2027-2042.

[18] F. Lutscher, E. Pachepsky and M.A. Lewis, *The effect of dispersal patterns on stream populations*, SIAM J. Appl. Math. 65 (2005), no. 4, 1305–1327

[19] J.D. Murray, Mathematical biology, Third edition. Interdisciplinary Applied Mathematics, 17. Springer-Verlag, New York, (2002).

[20] R.D. Nussbaum, *The radius of the essential spectrum.* Duke Math. J. 37 (1970), 473–478.

[21] K. Schumacher, *Travelling-front solutions for integro-differential equations. I.*, J. Reine Angew. Math. 316 (1980), 54–70.

[22] E. Zeidler, Nonlinear functional analysis and its applications. I. Fixed-point theorems. Springer-Verlag, New York, 1986.

[1] Departamento de Ingeniería Matemática, Universidad de Chile Blanco Encalada - 2120, 5 PISO, Santiago - Chile

[2] Centro de Modelamiento Matemático, UMI 2807 CNRS-Universidad de Chile, Av. Blanco Encalada 2120, Piso 7, Santiago - Chile

E-mail address: `jdavila@dim.uchile.cl`

E-mail address: `samartin@dim.uchile.cl`

Current address: [3] Max Planck Institut for Mathematics in the Sciences, Inselstrasse 22, D-04103 Leipzig, Deutschland

E-mail address: `coville@mis.mpg.de`