Recursive construction of periodic steady state for neural networks

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Abstract

We present a strategy in order to build neural networks with long steady-state periodic behavior. This strategy allows us to obtain $2^n$ nonequivalent neural networks of size $n$, when the equivalence relation is the usual one in dynamical systems. As a particular case, we build a neural network with $n$ neurons which realizes a cycle of period $2^n$.

1. Introduction

A neural network of size $n$ is a discrete dynamical system acting on $\{-1,1\}^n$, whose transition function, $F_A$, is given in terms of an $(n,n)$ real matrix $A=(a_{ij})$ as follows:

$$F_A(x) = \overline{\text{sgn}}(Ax); \quad (Ax)_i = \sum_{j=1}^{n} a_{ij}x_j, \quad i=1,\ldots,n,$$

$$\overline{\text{sgn}}: \mathbb{R}^n \to \{-1,1\}^n, \quad \overline{\text{sgn}}(y)_i = \text{sgn}(y)_i, \quad i=1,\ldots,n, \quad (1)$$

$$\text{sgn}(u) = \begin{cases} 1, & u \geq 0, \\ -1, & u < 0. \end{cases}$$

Neural networks were introduced by McCulloch and Pitts [6] to model some features of the neural system. The general definition for neural networks is given by taking $Ax-b$, with $b \in \mathbb{R}^n$, instead of $Ax$ in Eq. (1). Neural networks have been largely studied from the theoretical point of view by their wide applications in pattern recognition, memorization, learning, etc. For a survey in this subject see, for instance, [4, 7] for theoretical aspects and applications, respectively.
We focus our attention on \textit{reverberation neural networks} (RNN in the sequel), which are neural networks where each state of the system, after a finite number of steps, comes back to itself (hypercube permutations). In [1–3] the authors give relations between the rank of the matrix $A$ and the maximal period of the system defined by $F_A$. Later, in [8] it was proved that for RNN the rank of $A$ must be $n$. Also, in [9] it was proved that these neural networks are a subclass of the \textit{self-dual} neural networks which are defined by Eq. (1). Moreover, it was shown that for any RNN there are $2^n n!$ RNN such that their associated global transition functions differ. They are obtained by permutations and sign changes of rows of the weight matrix, $A$.

We study the question: \textit{how many reverberation neural networks have really different dynamics.}

For instance, in the example given in Fig. 1 in [9] the authors show that there are only four \textit{nonequivalent reverberation neural networks} of size 2, i.e., neural networks whose transition diagrams differ, and there are $2^2 2! = 8$ neural networks whose transition functions differ.

For neural networks of size 3 we know there are 14 \textit{nonequivalent reverberation neural networks} and $2^3 3! = 48$ reverberation neural networks whose transition functions differ. Our feeling is that the number of equivalence classes grows as an exponential function of $n$. This paper proves that there are $2^n n!$ \textit{nonequivalent reverberation neural networks} by recursively building them.

This result is proved in two parts: the recursive construction of neural networks and the proof that these neural networks are nonequivalents.

In the first part, we give a process which allows us recursively to build neural networks satisfying two properties: \textit{strictness} and \textit{variability} which are weak enough such that one can find a large number of neural networks satisfying them. This process is supported by Lemmas 1 and 2. Lemma 1 establishes a way of building from a signed function $f: \{-1,1\}^n \rightarrow \{-1,1\}$ another signed function $g: \{-1,1\}^n \rightarrow \{-1,1\}$ such that over a vector $(x,u)$ belonging to $\{-1,1\}^n \times \{-1,1\} \\setminus \{y_1,y_2,y_3,y_4\}$, $g(x,u) = f(x)$ and $g(y_i)$ for $i = 1,2,3,4$ is fixed by the construction.

For instance, from $f_a(y,z) = \text{sgn}(a \cdot (y,z))$, with $a = (1, -\frac{1}{2})$, Lemma 1 gives $g_b(y,z,u) = \text{sgn}(b \cdot (y,z,u))$ with $b = (1, -\frac{3}{8}, -\frac{3}{8})$, satisfying $g_b(y,z,u) = f_a(y,z)$ for every $y \neq z$ belonging to $\{-1,1\}^n \times \{-1,1\} \\setminus \{y_1,y_2,y_3,y_4\}$, $g_b(1,1,1) = -g_b(-1,1,1) = -1$ and $g_b(1,1,-1) = -g_b(-1,1,1) = 1$.

Also, applying Lemma 1 to $a = (-\frac{1}{2},1)$ we obtain $b = (-\frac{1}{2}, \frac{3}{8}, -\frac{3}{8})$, satisfying $g_b(y,z,u) = f_a(y,z)$ for every $y \neq z$, $g_b(1,1,1) = -g_b(-1,1,1) = -1$ and $g_b(1,1,-1) = -g_b(-1,1,1) = 1$. So, we can easily describe the dynamical evolution of $g$ in terms of those of $f$.

Lemma 2 gives a way of building signed functions which have an a priori desired behavior. For instance, we can obtain $c = (-1, -1, \frac{3}{2})$ and $d = (-1, -1, \frac{3}{2})$ satisfying: $g_d(x,y,u) = u$ and $g_c(y,z,u) = u$ for every $y \neq z$, $g_c(1,1,1) = -g_c(-1,1,1) = -1$ and $g_c(1,1,-1) = g_c(-1,1,1) = 1$.

From these two lemmas we give in Theorem 1, a recursive way for the construction of matrices. Given a matrix $A$ of size $n$ satisfying these two hypotheses, defined below,
we build two matrices $B$ and $C$ of size $n+1$ satisfying also these hypotheses. This
process will allow us to find a large number of neural networks. For instance, taking

$$A = \begin{pmatrix} a & \vdots \\ \alpha & 0 \end{pmatrix}.$$ 

Theorem 1 gives

$$B = \begin{pmatrix} b \\ \beta \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} a & 0 \\ \alpha & 0 \end{pmatrix}.$$ 

Observe that from $A$, which has four fixed points, we obtain $B$, with six fixed points
and one cycle of period 2, and $C$ with eight fixed points.

In the second part we define an equivalence relation on $P_n$, the set of bijective
functions from $\{-1, 1\}^n$ into $\{-1, 1\}^n$, and we build a function $\eta$ associating to each
element $\phi \in P_n$ a vector of size $2^n$ whose $i$th component gives the number of cycles of
period $i$ in the system defined by $\phi$. We prove that this function characterizes the
equivalence relation, i.e., two functions $F$ and $G$ are equivalent iff $\eta(F) = \eta(G)$. Hence,
we prove that the extensions $B$ and $C$ given in Theorem 1 define nonequivalent neural
networks by proving that $\eta(F_B)$ and $\eta(F_C)$ are different, where $F_B$ (resp. $F_C$) is the
transition function associated to $B$ (resp. $C$). For instance, $B$ and $C$ above are
nonequivalent because $B$ has a cycle of size 2 and $C$ has only fixed points. Later, we
prove that given two nonequivalent neural networks $A$ and $A'$ their extensions given
by Theorem 1 are also nonequivalent. This fact implies that by increasing the size of
the neural networks by one neuron, one can double the number of the nonequivalent
classes. That explains why we find $2^n$ nonequivalent neural networks.

As an application of the last results (see Corollary 2), we build a neural network $A$ of
size $n$ which has only one cycle of period $2^n$. There, the transition diagram for case
$n = 2$ is as follows:

$$\begin{pmatrix} -1 \\ -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

The diagram transition for case $n = 3$ is obtained by taking the previous diagram and
putting inside the inverse diagram, i.e., the diagram obtained by exchanging 1 by
$-1$ and $-1$ by 1. Under the first diagram we put four $-1$'s under the inverse diagram
we put four 1's. The final diagram is the following:

$$\begin{pmatrix} -1 \\ -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 \\ -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
The transition diagram for $n$ is obtained from the transition diagram for $n-1$ making a process as above. The technical difficulty that we solve consists in finding $A \in \mathbb{R}^n$ such that $F_A$ has a previous transition diagram.

2. Recursive construction of neural networks

The following properties are important in our construction and represent the possibility of modification for a vector.

**Definition 1.** Consider $a \in \mathbb{R}^n$. We say that:
(a) $a$ is **strict** if $\forall x \in \{-1,1\}^n$, $a \cdot x = \sum_{j=1}^{n} a_j x_j \neq 0$.
(b) $a$ is **variable** if $\exists I \in \{-1,1\}^n$, $a \cdot I < 0$ such that

$$\forall x \in \{-1,1\}^n, x \neq I, \quad a \cdot x < 0 \Rightarrow a \cdot x < a \cdot I.$$

Observe that for a vector $a$ satisfying (a) and (b) we have the scheme given in Fig. 1 which we adopt in order to give a more clear vision of the results. For a vector $a$ satisfying (a) and (b) one can have only $x$ such that $a \cdot x < 0$ or $a \cdot x > 0$ and then between $a \cdot I_a$ and $-a \cdot I_a$, in Fig. 1, there does not exist any value $a \cdot x$.

**Definition 2.** Previous definitions apply to a real $n \times n$ matrix $A$ by imposing that each row of $A$ satisfies them. More precisely, given a matrix $A$, we say:
(a') $A$ is **strict** if each row of $A$, $a^i$, for $i = 1, \ldots, n$, is strict.
(b') $A$ is **variable** if there exists a vector $I_A$ such that $a^i$ satisfies (b) with $I = I_A$, for every $i = 1, \ldots, n$.

When there exists a vector $I_A$ (resp. $I_a$) satisfying (b') (resp. b) we say that $A$ (resp. $a$) is $I_A$-variable (resp. $I_a$-variable).

In the sequel we will work with vectors and matrices verifying properties (a) and (b). So, we define

$$M^*_n(\mathbb{R}) = \{A: A \text{ is a strict variable } n \times n \text{ real matrix}\},$$

$$\mathbb{R}^*_n = \{a \in \mathbb{R}^n, \text{ is a strict variable vector}\}.$$

![Fig. 1. Scheme of the values of $a \cdot x$ where $x \in \{-1,1\}^n$ and $a$ is a strict variable vector.](image-url)
Definition 3. The transition function $F_a$ associated to vector $a \in \mathbb{R}^n$ is given by

$$F_a: \{-1,1\}^n \rightarrow \{-1,1\},$$

$x \rightarrow F_a(x) = \text{sgn}(a \cdot x)$. 

Observe that $F_A$ given in Eq. (1) can be written as follows:

$$F_A(x)_i = F_a(x) = \text{sgn}(a^i \cdot x), \quad i = 1, \ldots, n,$$

where $a^i$ is the $i$th row of $A$.

The following lemmas give the vector basic extensions. In these lemmas several technical details are given and in the sequel only its conclusions will be used.

In Lemma 1 we build a vector $a \in \mathbb{R}^n$, another vector $b \in \mathbb{R}^{n+1}$ such that the function $F_b$ is an extension of the function $F_a$ from $\mathbb{R}^n \setminus \{I_a, -I_a\}$ to $\mathbb{R}^{n+1} \setminus \{((\mu I_a, u) | u \in \{-1,1\}\}$ and such that $F_b$ over $\{((\mu I_a, u) | u \in \{-1,1\}\}$ takes values depending only on the $(n + 1)$th coordinate. In order to get a better understanding of Lemma 1 we show the meaning of the concepts used in the following example.

Consider the vector $a \in \mathbb{R}^2$ given by

$$a' = (1, -\frac{1}{2}). \quad \text{(2)}$$

Compute the values $a \cdot x$ for $x \in \{-1,1\}^2$. Since $a \cdot x = -a \cdot (-x)$ we get

$$a \cdot (1, 1) = 1 - \frac{1}{2} = \frac{1}{2}, \quad a \cdot (-1, 1) = -\frac{1}{2},$$

$$a \cdot (-1, 1) = -1 - \frac{1}{2} = -\frac{3}{2}, \quad a \cdot (1, -1) = \frac{3}{2}.$$

Clearly $a$ is strict. Let $I_a = (-1, -1)$. Then since $-\frac{3}{2} < -\frac{1}{2} < 0 < \frac{1}{2} < \frac{3}{2}$ a is $I_a$-variable. Let $D_a^-(a), D_a^+(a)$ be given by

$$D_a^-(a) = \{x \in \{-1,1\}^n | a \cdot x < 0, x \neq I_a\}, \quad \text{(3)}$$

$$D_a^+(a) = \{x \in \{-1,1\}^n | a \cdot x > 0, x \neq I_a\}. \quad \text{(4)}$$

In this case, $n = 2$, $I_a = (-1, -1)$ and $a' = (1, -\frac{1}{2})$. Then $D_a^-(a) = \{(1, 1)\}$, $D_a^+(a) = \{(1, -1)\}$. Let $h_a$ be the maximum value in $D_a^-(a)$ given by

$$h_a = \max \{a \cdot x : x \in D_a^-(a)\}. \quad \text{(5)}$$

Then $h_a = -\frac{3}{2}$. Let $\delta > 0$ be such that $2a \cdot I_a = -1 - \delta < -\frac{1}{2} = a \cdot I_a$ and $(h_a + a \cdot I_a)/2 = -1 - \delta < -\frac{1}{2} = a \cdot I_a$. Taking $\delta = \frac{3}{4}$ and $v = (I_a)_2 = -1$ we define $b \in \mathbb{R}^3$ by

$$b_1 = a_1 = 1, \quad b_2 = a_2 + \frac{v}{2} = -\frac{1}{2} - \frac{3}{8} = -\frac{7}{8}, \quad b_3 = -\frac{4}{2} = -\frac{3}{8}.$$

Then $b$ is given by

$$b' = (1, -\frac{7}{8}, -\frac{3}{8}) \quad \text{(6)}$$

In Lemma 2 we build a vector $a \in \mathbb{R}^3$, another vector $b \in \mathbb{R}^{n+1}$ such that the function $F_b$ is an extension of the function $F_a$ from $\mathbb{R}^3 \setminus \{I_a, -I_a\}$ to $\mathbb{R}^{n+1} \setminus \{((\mu I_a, u) | u \in \{-1,1\}\}$ and such that $F_b$ over $\{((\mu I_a, u) | u \in \{-1,1\}\}$ takes values depending only on the $(n + 1)$th coordinate. In order to get a better understanding of Lemma 2 we show the meaning of the concepts used in the following example.

Consider the vector $a \in \mathbb{R}^3$ given by

$$a' = (1, -\frac{1}{2}, -\frac{1}{2}). \quad \text{(2)}$$

Compute the values $a \cdot x$ for $x \in \{-1,1\}^3$. Since $a \cdot x = -a \cdot (-x)$ we get

$$a \cdot (1, 1, 1) = 1 - \frac{1}{2} - \frac{1}{2} = \frac{1}{2},$$

$$a \cdot (-1, 1, 1) = -1 - \frac{1}{2} - \frac{1}{2} = -\frac{3}{2},$$

$$a \cdot (-1, -1, 1) = -1 - \frac{1}{2} + \frac{1}{2} = -1,$$
Fig. 2. Scheme of the values of \((1, -1/2) \cdot x\) where \(x \in \{-1, 1\}^2\).

and \(b \cdot (x, u)\) for \((x, u) \in \{-1, 1\}^2 \times \{-1, 1\}\) is given by (see Fig. 2)

\[
\begin{align*}
  b \cdot (1, 1, 1) &= -b \cdot (-1, -1, -1) = -\frac{1}{2}, \\
  b \cdot (-1, 1, 1) &= -b \cdot (1, -1, -1) = -\frac{3}{4}, \\
  b \cdot (1, -1, 1) &= -b \cdot (-1, 1, -1) = \frac{3}{2}, \\
  b \cdot (1, 1, -1) &= -b \cdot (-1, -1, 1) = \frac{1}{2}.
\end{align*}
\]

For a vector \(I \in \mathbb{R}^n\) and an element \(v \in \mathbb{R}\) we denote by \((I, v)^t\) the extension of \(I\) from \(\mathbb{R}^n\) into \(\mathbb{R}^{n+1}\) whose \((n+1)\)th coordinate is \(v\).

So, \(b\) is \textit{strict} and taking \(I_b = (-I_a, 1) = (1, 1, 1)\) one obtains that \(b \cdot I_b = -\frac{1}{2}\) and \(h_b = \frac{1}{2}\) and then \(b\) is \(I_b\)-\textit{variable}. Moreover, \(F_b\) and \(F_a\) are related by

\[
\begin{align*}
  F_b(-1, 1, 1) &= F_b(-1, 1, -1) = F_a(-1, 1) = -1, \\
  F_b(1, -1, 1) &= F_b(1, 1, -1) = F_a(1, -1) = 1
\end{align*}
\]

and

\[
F_b(I_a, 1) = -1 = F_b(- I_a, 1), \quad F_b(I_a, -1) = 1 = F_b(- I_a, -1).
\]

So,

\[
F_b(x, u) = F_a(x), \quad x \neq I_a, \quad x \neq -I_a
\]

and

\[
F_b(\mu I_a, u) = -u, \quad \mu, u \in \{-1, 1\}.
\]

Taking

\[
\tilde{a}^t = (-\frac{1}{2}, 1),
\]

we deduce that \(I_{\tilde{a}} = (-1, -1), \quad h_{\tilde{a}} = -\frac{3}{2}\) and \(\tilde{a} \cdot I_{\tilde{a}} = -\frac{1}{2}\). For \(\tilde{a}\) we define vector \(\tilde{b}\) by

\[
\tilde{b}^t = (-\frac{1}{2}, 1 + (-1) \frac{3}{8}, -\frac{3}{8}) = (-\frac{1}{2}, \frac{3}{8}, -\frac{3}{8}).
\]

It is easy to see that \(\tilde{b}\) is \textit{strict}, \(I_{\tilde{b}} = (-I_a, v) = (1, 1, 1)\)-\textit{variable} and that \(F_{\tilde{b}}\) satisfies Eqs. (7) and (8) exchanging \(a\) by \(\tilde{a}\). The generalization of this result is given in Lemma 1.

**Lemma 1.** Let \(a \in \mathbb{R}_{-}^n\). Then, there exists \(b \in \mathbb{R}_{-}^{n+1}\) satisfying

(a) \(I_b = (-I_a, 1),\)
(b) \(\forall (x, u) \in \{-1, 1\}^n \times \{-1, 1\}, \ x \neq -I_a, \ x \neq I_a, \ F_b(x, u) = F_a(x),\)
(c) \(\forall \mu, u \in \{-1, 1\} \ F_b(\mu I_a, u) = -u.\)
Proof. Let $D_n^-(a), D_n^+(a)$ be given as in Eqs. (3) and (4). Since $a$ is a strict vector we have the following equivalence:

$$x \in \{-1, 1\}^n \iff x \in \{-I_a, I_a\} \lor x \in D_n^-(a) \lor x \in D_n^+(a).$$

(11)

Let $h_a$ be the maximum value in $D_n^-(a)$ given in Eq. (5). Since $a$ is $I_a$-variable we have that $h_a < a \cdot I_a < 0$ and then (see Fig. 3)

$$2a \cdot I_a < a \cdot I_a \quad \text{and} \quad \frac{h_a + a \cdot I_a}{2} < a \cdot I_a,$$

(12)

so there exists $\delta > 0$ such that

$$2a \cdot I_a < -\delta < a \cdot I_a \quad \text{and} \quad \frac{h_a + a \cdot I_a}{2} < -\delta,$$

(13)

which is equivalent to

$$-(a \cdot I_a + \delta) < 0, \quad h_a + \delta < -(a \cdot I_a + \delta) \quad \text{and} \quad a \cdot I_a < -(a \cdot I_a + \delta).$$

(14)

Observe that in Fig. 3 we suppose that $(h_a + a \cdot I_a)/2 > 2a \cdot I_a > h_a$, which is not the general case. From definition of $h_a$ and Eq. (14) we have for $x \in D_n^-(a)$ that

$$a \cdot x + \delta \leq h_a + \delta < -(a \cdot I_a + \delta) < 0,$$

(15)

hence $\delta < -a \cdot x$. Since $x \in D_n^-(a)$ iff $-x \in D_n^+(a)$ we obtain

$$\forall x \in D_n^-(a) \cup D_n^+(a), \quad |a \cdot x| > \delta.$$  

(16)

Define $b \in \mathbb{R}^{n+1}$ by

$$b_i = a_i, \quad i = 1, \ldots, n-1, \quad b_n = a_n + v \frac{2}{n}, \quad b_{n+1} = -\frac{2}{n},$$

where $(I_a)_n = v$. It is clear that for $(x, u) \in \{-1, 1\}^n \times \{-1, 1\}$ we have

$$b \cdot (x, u) = a \cdot x + (v x_n - u) \frac{\delta}{2}$$

(17)

and then

$$|b \cdot (x, u) - a \cdot x| \leq \delta.$$  

(18)
We shall prove that $b$ is variable. For that we want to find $(x,u)$ satisfying $b \cdot (x,u) < 0$. Let $x \in D_n^-(a) \cup D_n^+(a)$, then from Eqs. (16) and (18) we have
\begin{equation}
a \cdot x > 0 \Rightarrow b \cdot (x,u) \geq a \cdot x - \delta > 0
\end{equation}
and
\begin{equation}
a \cdot x < 0 \Rightarrow b \cdot (x,u) \leq a \cdot x + \delta < 0.
\end{equation}
Let $x = \mu I_a$ with $\mu = -1, 1$. Since $(I_a)_n = v$, from Eq. (17) we have
\begin{equation}
b \cdot (I_a,1) = a \cdot I_a + \left(\nu \nu \mu - 1 \right) \frac{\delta}{2} = a \cdot I_a + (\mu - 1) \frac{\delta}{2},
\end{equation}
so from Eqs. (14) and (21) we obtain
\begin{align*}
\mu = 1 & \Rightarrow b \cdot (I_a,1) = a \cdot I_a < -(a \cdot I_a + \delta) < 0, \\
\mu = -1 & \Rightarrow b \cdot (-I_a,1) = -a \cdot I_a + (1 - 1) \frac{\delta}{2} = -(a \cdot I_a + \delta) < 0,
\end{align*}
i.e., $b \cdot (I_a,1) < 0$ for $\mu = -1, 1$. Let $I_b = (-I_a,1)$. Then from Eq. (23) we get $b \cdot I_b < 0$ and applying Eqs. (19), (20), (22) and (23) we obtain the following equivalence:
\begin{equation}
b \cdot (x,u) < 0 \text{ and } (x,u) \neq I_b \text{ iff } a \cdot x < 0 \land x \neq I_a \lor (x,u) = (I_a,1).
\end{equation}
Let $(x,u) \in \{-1,1\}^{n+1}$ such that $b \cdot (x,u) < 0 \land (x,u) \neq I_b$. From Eq. (24) there are only two possibilities for $(x,u)$. For the first one, i.e., $a \cdot x < 0$ and $x \neq I_a$, we know from Eqs. (15), (20) and (23) that
\begin{equation}
b \cdot (x,u) \leq a \cdot x + \delta < -(a \cdot I_a + \delta) = b \cdot I_b.
\end{equation}
For the second one, from Eq (22) we get that
\begin{equation}
b \cdot (I_a,1) = a \cdot I_a < b \cdot I_b,
\end{equation}
which proves that $b$ is $I_b$-variable. Observe that the inequalities in Eqs. (19), (20), (22) and (23) are strict, so
\begin{equation}
\forall (x,u) \in \{-1,1\}^{n+1}, \quad b \cdot (x,u) < 0 \lor b \cdot (x,u) > 0,
\end{equation}
which says that $b$ is strict.

Finally, from Eqs. (19), (20), (22) and (23) we get
\begin{align*}
\forall (x,u) & \in \{-1,1\}^{n+1}, \quad x \in D_n^-(a) \cup D_n^+(a), \\
F_b(x,u) & = \text{sgn}(b \cdot (x,u)) = \text{sgn}(a \cdot x) = F_a(x)
\end{align*}
and
\begin{align*}
\forall u, \mu & \in \{-1,1\}, \quad F_b(\mu I_a,u) = \text{sgn}(b \cdot (\mu I_a,u)) = \text{sgn}(-u) = -u.
\end{align*}

In the next lemma we build two vectors $c$ and $d$ in $\mathbb{R}_*^{n+1}$. Vector $d$ is such that the function $F_d$ is the projection over the $(n+1)$th coordinate. Vector $c$ defines the function $F_c$ being the projection of the $(n+1)$th coordinate from
\[ \mathbb{R}^{n+1} \setminus \{(\mu I_a, u) | \mu, u \in \{-1, 1\}\} \] into \{-1, 1\} and it considers only the sign of the \(n\)th coordinate of \((\mu I_a, u)\) for \(\mu, u \in \{-1, 1\}\).

In order to show how the proof of Lemma 2 proceeds, we give an example: Let \(I_a = (-1, -1)\). We define
\[
c = (-1, -1, 2 - \frac{1}{2}) = (-1, -1, \frac{3}{2})
\]
and
\[
d = (-1, -1, 2 - \frac{1}{2}) = (-1, -1, \frac{3}{2}).
\]

Then \(c \cdot (x, u)\) for \((x, u) \in \{-1, 1\}^2 \times \{-1, 1\}\) is given by
\[
c \cdot (-1, -1, -1) = -c \cdot (1, 1, 1) = \frac{1}{3}, \quad c \cdot (-1, -1, 1) = -c \cdot (1, 1, -1) = \frac{1}{3},
\]
\[
c \cdot (-1, 1, 1) = -c \cdot (1, -1, 1) = \frac{1}{3}, \quad c \cdot (-1, 1, -1) = -c \cdot (1, -1, -1) = -\frac{1}{3},
\]
so \(c\) is strict, \(I_c = (-I_a, 1) = (1, 1, -1)\)-variable and satisfies \(F_c(x, u) = u\), \(x \neq I_a, x \neq -I_a\). \(F_c(-I_a, -1) = -1\) and \(F_c(-I_a, 1) = -1\), i.e., \(F_c(\mu I_a, u) = \mu\).

On the other hand, \(d \cdot (x, u)\) is given by
\[
d \cdot (-1, -1, -1) = -d \cdot (1, 1, 1) = -\frac{1}{3}, \quad d \cdot (-1, -1, 1) = -d \cdot (1, 1, -1) = \frac{1}{3},
\]
\[
d \cdot (-1, 1, 1) = -d \cdot (1, -1, 1) = \frac{1}{3}, \quad d \cdot (-1, 1, -1) = -d \cdot (1, -1, -1) = -\frac{1}{3},
\]
so \(d\) is strict, \((I_a, -1)\)-variable and \(F_d(x, u) = u\).

**Lemma 2.** For \(I_a \in \mathbb{R}^n\) there exist \(c \in \mathbb{R}^{n+1}\) such that
\[(a) \quad I_c = (-I_a, 1) = (I_a, -1), \]
\[(b1) \quad \forall(x, u) \in \{-1, 1\}^n \times \{-1, 1\}, x \neq I_a, x \neq -I_a, F_c(x, u) = u, \]
\[(b2) \quad \forall(a, u) \in \{-1, 1\}^n, F_c(\mu I_a, u) = \mu, \]
\[(c) \quad \forall(x, u) \in \{-1, 1\}^n \times \{-1, 1\}, F_d(x, u) = u.\]

**Proof.** The construction of \(c\) and \(d\) is very similar. So, we give this construction in only a vector \(e(r)\) which will be appropriately evaluated in order to obtain \(c\) and \(d\). Let \(e(r) = (I_a, (n-r)/2), |r| = 1\) belongs to \(\mathbb{R}^{n+1}\). For \((x, u) \in \{-1, 1\}^n \times \{-1, 1\}\) we have
\[
e(r) \cdot (x, u) = x \cdot I_a + u \left(\frac{n-r}{2}\right). \tag{27}
\]
It is easy to see that \(x \neq \mu I_a, \mu = -1, 1\) is equivalent to \(-n+2 \leq x \cdot I_a \leq n-2\), which applied to Eq. (27) implies
\[
u = 1, \quad e(r) \cdot (x, 1) \geq -n+2 + n - \frac{r}{2} = 2 - \frac{r}{2} > 1, \tag{28}
\]
\[
u = -1, \quad e(r) \cdot (x, -1) \leq -n+2 + n - 2 = -2 + \frac{r}{2} < -1, \tag{29}
\]
i.e.,
\[ \text{sgn}(e(r) \cdot (x,u)) = u \] when \( x \neq \mu I_a, \mu = -1, 1. \)

Let \( x = \mu I_a, \mu = 1, -1. \) Then
\[ e(r) \cdot (\mu I_a, u) = \mu I_a \cdot I_a + u \left( n - \frac{r}{2} \right) = (\mu + u)n - \frac{ur}{2}. \]

So, \( |e(r) \cdot (\mu I_a, u)| = |(\mu + u)n - \frac{ur}{2}| \geq |\mu + u|n - |ur/2| = |\mu + u|n - \frac{1}{2}| > 0. \) Hence from Eqs. (28) and (29) \( e(r) \) is strict.

We prove that \( e(r) \) is \( I_{e(r)} \)-variable with \( I_{e(r)}^r = r(-I_a, 1). \) Compute the value \( e(r) \cdot I_{e(r)}: \)
\[ e(r) \cdot I_{e(r)} = \left( (-r + r)n + \frac{-rr}{2} \right) = -\frac{1}{2}; \]
moreover,
\[ e(r) \cdot (-I_a, -1) = \left( -2n + \frac{r}{2} \right) = -2n + \frac{r}{2} < -1. \]

Since \( r(-I_a, 1) \neq (-I_a, -1) \) we obtain
\( (x, u) \neq I_{e(r)} \) and \( e(r) \cdot (x, u) < 0 \) iff \( x \neq I_a, x \neq -I_a \) and \( u = -1 \)
or \( (x, u) = (-I_a, -1) \)
and then for \( e(r) \cdot (x, u) < 0 \) we get
\( (x, u) \neq I_{e(r)} \Rightarrow e(r) \cdot (x, u) < -1 < e(r) \cdot I_{e(r)} = -\frac{1}{2}. \)

So, \( e(r) \) is \( I_{e(r)} \)-variable.

Taking \( c = e(1) \) and \( d = e(-1) \) it is easy to see that (a), (b) and (c) are satisfied. \( \square \)

The extension for a matrix \( A \) is given in the following theorem. As an example of the construction consider the real matrix \( A \) given by
\[ A = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} a \\ \bar{a} \end{pmatrix}, \]
where \( a \) and \( \bar{a} \) are given in Eqs. (2) and (9). Then from the analysis for \( a \) and \( \bar{a} \), \( A \) is strict and \( I_A = (-1, 1) \)-variable. Consider \( B \) given by
\[ B = \begin{pmatrix} b \\ \bar{b} \\ c \end{pmatrix} = \begin{pmatrix} 1 & -\frac{7}{8} & -\frac{3}{8} \\ -\frac{1}{2} & \frac{7}{8} & -\frac{3}{8} \\ -1 & -1 & \frac{3}{2} \end{pmatrix}, \]
where \( B, \bar{b} \) and \( c \) were constructed in Eqs. (6), (10) and (25). Then \( B \) is strict and \((1, 1, 1)\)-variable. Moreover, for \( x \neq I_A \), \( x \neq -I_A \)
\[ F_B(x, u) = (F_b(x, u), F_b(x, u), F_c(x, u))^t = (F_a(x), F_a(x), u)^t = (F_A(x), u)^t \]
and
\[ F_B(\mu I_A, u) = (F_b(\mu I_A, u), F_b(\mu I_A, u), F_c(\mu I_A, u))^T = (-u, -u, \mu)^T = (-ue_2, \mu)^T. \]

where \( e_2^T = (1, 1) \). Now, let
\[ C = \begin{pmatrix}
    a & 0 \\
    \bar{a} & 0 \\
    d
\end{pmatrix} = \begin{pmatrix}
    1 & -\frac{1}{2} & 0 \\
    -\frac{1}{2} & 1 & 0 \\
    -1 & -1 & \frac{\delta}{2} \\
\end{pmatrix},
\]

where \( \delta \) is given by Eq. (26). Then \( C \) is strict and \((I_A, -1)\)-variable. Moreover,
\[ F_C(x, u) = (F_d(x), F_d(x), F_d(x, u))^T = (F_d(x), u)^T. \]

The last construction is generalized for any matrix in \( M_n^*(\mathbb{R}) \) in the following theorem.

**Theorem 1.** For \( A \in M_n^*(\mathbb{R}) \), there exist \( B \) and \( C \) in \( M_{n+1}^*(\mathbb{R}) \) such that
\[ (4) \ I_b^T = (-I_A, 1), \quad I_c^T = (I_A, -1), \]
\[ (5) \ \forall (x, u) \in \{-1, 1\}^n \times \{-1, 1\}, \quad F_b(x, u) = (F_b(x), u)^T, \]
\[ (6) \ \forall (x, u) \in \{-1, 1\}^n \times \{-1, 1\}, \quad x \neq -I_A, \quad x \neq I_A, \quad F_b(x, u) = (F_b(x), u)^T, \]
\[ (7) \ \forall (x, u) \in \{-1, 1\}^n \times \{-1, 1\}, \quad x \neq -I_A, \quad x \neq I_A, \quad F_b(x, u) = (F_b(x), u)^T, \]
\[ (8) \ \forall (x, u) \in \{-1, 1\}^n \times \{-1, 1\}, \quad x \neq -I_A, \quad x \neq I_A, \quad F_b(x, u) = (F_b(x), u)^T, \]
\[ (9) \ \forall (x, u) \in \{-1, 1\}^n \times \{-1, 1\}, \quad x \neq -I_A, \quad x \neq I_A, \quad F_b(x, u) = (F_b(x), u)^T, \]
\[ (10) \ \forall (x, u) \in \{-1, 1\}^n \times \{-1, 1\}, \quad x \neq -I_A, \quad x \neq I_A, \quad F_b(x, u) = (F_b(x), u)^T, \]
\[ (11) \ \forall (x, u) \in \{-1, 1\}^n \times \{-1, 1\}, \quad x \neq -I_A, \quad x \neq I_A, \quad F_b(x, u) = (F_b(x), u)^T. \]

where \( e_n^T = (1, \ldots, 1) \in \mathbb{R}^n. \)

**Proof.** Construction of matrix \( B \): Since \( A \in M_n^*(\mathbb{R}) \) we know that each row \( a_i \), \( i = 1, \ldots, n \), belongs to \( \mathbb{R}_n^* \). Applying Lemma 1 to each row we find vectors \( b_i \in \mathbb{R}_{n+1}^* \) which are \((I_A, -1)\)-variables, satisfying
\[ \forall (x, u) \in \{-1, 1\}^n \times \{-1, 1\}, \quad x \neq -I_A, \quad x \neq I_A, \quad F_b(x, u) = F_{a_i}(x) \quad \text{and} \quad F_b(x, u) = -u. \]

By applying Lemma 2, for \( I = \lambda I_A \), we obtain \( c \in \mathbb{R}_{n+1}^* \) \((I_A, 1)\)-variable, such that
\[ \forall (x, u) \in \{-1, 1\}^n \times \{-1, 1\}, \quad x \neq -I_A, \quad x \neq I_A, \quad F_c(x, u) = u, \quad \text{and} \quad F_c(x, u) = \mu. \]

Define \( B^* = (b^1, b^2, \ldots, b^n, c) \). Since each \( b^i \), for \( i = 1, \ldots, n \), belongs to \( \mathbb{R}_{n+1}^* \) with \( I_{b^i} = (-I_A, 1) \) we know that \( B \in M_{n+1}^*(\mathbb{R}) \), and from Lemma 1 and conclusion (b) of Lemma 2, \( B \) verifies properties (a) and (c) of the theorem.

**Construction of matrix \( C \):** Let \( d \) be the vector given by Lemma 2 which is \((I_A, -1)\)-variable. Let \( C \) be defined by
\[ C_{ij} = a_{ij}, \quad 1 \leq i, j \leq n, \quad C_{j, n+1} = 0, \quad 1 \leq j \leq n, \quad C_{n+1, j} = d_j, \quad 1 \leq j \leq n+1. \]

Since \( d \in \mathbb{R}_{n+1}^* \) with \( I_d = (-I_A, 1) \), \( C \in M_{n+1}^* \). Moreover,
\[ C = \begin{pmatrix} x \\ u \\ d \cdot u \end{pmatrix} = \begin{pmatrix} Ax \\ d \cdot u \end{pmatrix}. \]
and then $F_C$ is given by $(F_A(x), F_d(u))^t = (F_A(x), u)$. So, $C$ satisfies (a) and (b) in the theorem. □

3. Nonequivalent neural networks

Consider the set $P_n$ of the bijective functions on $\{-1, 1\}^n$. The following property is shown in [9] for $F \in P_n$:

$$\forall x \in \{-1, 1\}^n \exists s \in \mathbb{N}, \quad F^s(x) = x.$$  \hspace{1cm} (31)

We define the cycle of $x$ by $F$, $O_F(x)$, for $F \in P_n$ by

$$O_F(x) = \langle x, F(x), \ldots, F^{T_F^{-1}}(x) \rangle,$$

where $T_F$ is the first integer such that $F^{T_F}(x) = x$. $T_F$ is called the period of the cycle $O_F(x)$. We say that $y \sim O_F(x)$ iff there exists $s \in \mathbb{N}$ such that $F^s(x) = y$. Taking

$$A = \begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & -\frac{1}{2} \end{pmatrix}$$

we have

$$O_{F_A}(1, 1) = \langle (1, 1) \rangle, \quad O_{F_A}(-1, 1) = \langle (-1, -1) \rangle$$

and

$$O_{F_A}(1, -1) = \langle (1, -1), (-1, -1) \rangle.$$

Then

$$T_{(1, 1)}^{F_A} = 1 = T_{(-1, 1)}^{F_A}, \quad T_{(-1, 1)}^{F_A} = T_{(1, -1)}^{F_A} = 2.$$

In order to show the power of the construction given in Section 2 it is necessary to specify when two neural networks have different dynamics. For that we define the following equivalence relation: Given $F$ and $G$ in $P_n$, we say that $F$ is equivalent to $G$ iff there exists a function $\Phi$ on $P_n$ such that

$$\forall x \in \{-1, 1\}^n, \quad F(\Phi(x)) = \Phi(F(x)).$$  \hspace{1cm} (32)

This definition does not permit easily to prove that our construction builds nonequivalent neural networks. For that, given a function $F \in P_n$, we define the characteristic of $F$ by a vector $\eta(F)$ in $\mathbb{N}^2$, such that its $i$th component gives the cycle numbers of period $i$ of $F$ and we prove the following lemma.

**Lemma 3.** Two functions $F$ and $G$ in $P_n$ are equivalent iff $\eta(F) = \eta(G)$.

**Proof.** ($\Rightarrow$) We prove the following equivalence for $\Phi$ satisfying Eq. (32):

$$C^F = \langle x, F(x), \ldots, F^{L-1}(x) \rangle$$

is a cycle for $F$ iff $C^G$

$$= \langle \Phi(x), \Phi(F(x)), \ldots, \Phi(F^{L-1}(x)) \rangle$$

is a cycle for $G$. 

Indeed, since $F$ and $G$ are equivalent we have that
\[ \Phi(F^i(x)) = G^i(\Phi(x)) \quad \text{for } i = 0, \ldots, L - 1 \]
and then Eq. (33) is true. Hence for each cycle of size $L$ of $F$ we have a cycle of size $L$ for $G$ and conversely for each cycle of size $L$ of $G$ we have a cycle of size $L$ for $G$ with which $\eta(F) = \eta(G)$.

$(\Leftarrow)$ since $F$ and $G$ belongs to $P_n$, a vector $x \in \{-1, 1\}^n$ can belong to only one cycle.

Let $C^i_j, j = 1, \ldots, n_i$, be the different cycles of size $i$ for $F$ and $G$, respectively. We define the function $\Phi$ associating $C^i_j$ to $\beta^i_j$ as follows: Let $C^i_j = \langle x, F(x), \ldots, F^{i-1}(x) \rangle$ and $\beta^i_j = \langle y, G(y), \ldots, G^{i-1}(y) \rangle$, then we define $\Phi$ by
\[ \Phi(G^k(y)) = F^k(x), \quad 1 \leq k \leq i - 1, \quad \Phi(y) = x. \]

Making this process for any $j$ and any $i$ we define completely $\Phi$ satisfying $\Phi F = G \Phi$. $\square$

Definition 4. We say that a real matrix $A$ is a reverberation neural network if $F_A$ belongs to $P_n$.

Proposition 1. Let $A \in M_n^*(\mathbb{R})$ be a reverberation neural network. Then $B$ and $C$ given in Theorem 1 are reverberation neural networks and the periods of their cycles are determined in terms of the periods of the cycles of $F_A$ as follows:

\[ \forall (x, u) \in \{-1, 1\}^n \times \{-1, 1\}, \quad T_{(x, u)}^{F_B} = T_x^{F_A}, \quad (34) \]

\[ \forall (x, u) \in \{-1, 1\}^n \times \{-1, 1\}, \quad I_A - I_A \notin O_{F_A}(x), \quad T_{(x, u)}^{F_B} = T_x^{F_A}, \quad (35) \]

\[ \text{if } \mu I_A \in O_{F_A}(x) \text{ for some } \mu = -1, 1 \text{ then } T_{(x, u)}^{F_B} = T_{(\mu I_A, u)}^{F_B}, \quad (36) \]

where
\[ T_{(\mu I_A, u)}^{F_B} = \begin{cases} 2T_x^{F_A} & \text{if } e_n \in O_{F_A}(I_A), \\ 2T_x^{F_A} & \text{if } u = -\mu \text{ and } e_n \notin O_{F_A}(I_A), \\ T_x^{F_A} & \text{if } u = \mu \text{ and } e_n \notin O_{F_A}(I_A). \end{cases} \quad (37) \]

Proof. Before giving the proof we analyze our example. From the definition of $B$ and $C$ it is easy to see that
\[ O_{F_B}(1, -1, -1) = O_{F_B}(-1, 1, -1) = \langle (1, -1, -1), (1, 1, -1) \rangle, \]
\[ O_{F_B}(-1, 1, 1) = O_{F_B}(1, -1, 1) = \langle (1, 1, 1), (1, -1, 1) \rangle, \]
\[ O_{F_B}(-1, -1, 1) = -O_{F_B}(1, 1, -1) = \langle (-1, -1, 1), (-1, 1, 1) \rangle, \]
\[ O_{F_B}(1, 1, 1) = O_{F_B}(-1, -1, -1) = \langle (1, 1, 1), (1, 1, 1) \rangle. \]
and
\[
O_F(1, -1, -1) = O_F(-1, 1, -1) = \langle (1, -1, -1), (-1, 1, -1) \rangle,
\]
\[
O_F(-1, 1, -1) = O_F(1, -1, 1) = \langle (-1, 1, 1), (1, -1, 1) \rangle,
\]
\[
O_F(-1, -1, 1) = -O_F(1, 1, -1) = \langle (-1, -1, 1) \rangle,
\]
\[
O_F(1, 1, 1) = -O_F(-1, -1, -1) = \langle (1, 1, 1) \rangle.
\]
and then
\[
\forall (x, u) \in \{-1, 1\}^2 \times \{-1, 1\}, \quad T^{F_c}_{\{x, u\}} = T^{F_A}_{x},
\]
\[
\forall x \in \{-1, 1\}^2, \quad x \neq I_A, \quad x \neq -I_A, \quad T^{F_B}_{\{x, u\}} = T^{F_A}_{x},
\]
\[
T^{F_B}_{(I_A, 1)} = T^{F_B}_{(-I_A, -1)} = 1 = T^{F_A}_{I_A}
\]
and
\[
T^{F_B}_{(-I_A, -1)} = 2 = 2T^{F_A}_{I_A}.
\]

Note that we are in the case \(e \notin O_{F_B}(I_A)\), \((I_A, 1)\) and \((-I_A, -1)\) satisfy the condition \(u = \mu\) and \((-I_A, 1)\) and \((I_A, -1)\) satisfy the condition \(u = -\mu\). This proves the proposition in our example.

Now we give the general proof. First, we prove that \(B\) and \(C\) are reverberation neural networks. Suppose that \(F_c(x, u) = F_c(x', u')\). Since \(F_c(x, u) = (F_A(x), u)\) we have that \(u = u'\) and \(F_A(x) = F_A(x')\). But \(A\) is a reverberation neural network, so \((x, u) = (x', u')\) and \(C\) is a reverberation neural network. Now, suppose that \(F_B(x, u) = F_B(x', u')\). Then if \(x \neq \mu I_A\) we proceed as above. When \(x = \mu I_A\) we have \(F_B(\mu x, u) = (-ue, u)\). Since \(A\) is a reverberation neural network, \(F_A(y) = -ue\) only for \(y = \mu I_A\). Then \(x' = \mu I_A\) and from \((-ue, u') = (-ue, u')\) we conclude that \((x, u) = (x', u')\) and \(B\) is a reverberation neural network.

Properties (34) and (35) follow from the fact that
\[
\forall k \in \mathbb{N}, \quad F^k_B(x, u) = (F^k_B(x), u) \quad \text{and} \quad F^k_B(x, u) = (F^k_B(x), u) \quad \text{when} \quad F^k_A(x) \neq -I_A, I_A.
\]

When \(\mu I_A \in O_{F_B}(x)\) we have that
\[
O_{F_c}(x, u) = \langle (x, u), \ldots, (z, t), (\mu I_A, u), (-ue, \mu), \ldots, (y, w) \rangle,
\]
and since \(F_B \in P_{n+1}, \ O_{F_B}(\mu I_A, u)\) is given by
\[
O_{F_B}(\mu I_A, u) = \langle (\mu I_A, u), (-ue, \mu), \ldots, (y, w)(x, u), \ldots, (z, t) \rangle.
\]

Observe the structure of \(O_{F_B}(\mu I_A, u)\). Suppose that \(e \in O_{F_A}(I_A)\), then since \(F_A \in P_n\) the following sequence of transition is true:

\[
I_A \rightarrow e \rightarrow \ldots \rightarrow I_A \rightarrow e \rightarrow \ldots \rightarrow I_A.
\]
and then from Theorem 1

\[(\mu I_A, \mu) \rightarrow (-\mu I_A, \mu) \rightarrow \cdots \rightarrow (-\mu I_A, \mu) \rightarrow (\mu I_A, \mu) \rightarrow \cdots \rightarrow (\mu I_A, \mu),\]

\[(-\mu I_A, -\mu) \rightarrow (\mu I_A, -\mu) \rightarrow \cdots \rightarrow (\mu I_A, \mu) \rightarrow (\mu I_A, -\mu) \rightarrow \cdots \rightarrow (\mu I_A, \mu),\]

i.e., \(T_{(\mu I_A, \mu)}^F = T_{(\mu I_A, \mu)}^F = T_{(\mu I_A, \mu)}^F = 2T_{I_A}^F.\)

If \(e \notin O_{F_A}(I_A)\) then

\[(\mu I_A, \mu) \rightarrow (\mu I_A, \mu) \rightarrow \cdots \rightarrow (\mu I_A, \mu),\]

i.e., \(T_{(\mu I_A, \mu)}^F = T_{(\mu I_A, \mu)}^F\), moreover,

\[(\mu I_A, -\mu) \rightarrow (\mu I_A, -\mu) \rightarrow \cdots \rightarrow (\mu I_A, \mu),\]

\[(-\mu I_A, \mu) \rightarrow (-\mu I_A, \mu) \rightarrow \cdots \rightarrow (-\mu I_A, \mu),\]

i.e., \(T_{(\mu I_A, -\mu)}^F = 2T_{(\mu I_A, \mu)}^F\) and we have the conclusions. 

Observe that in our example we have \(\eta(F_A) = (2, 2, 0, 0), \eta(F_B) = (2, 3, 0, 0, 0, 0, 0, 0)\) and \(\eta(F_C) = (4, 2, 0, 0, 0, 0, 0, 0)\), which motivates the following corollary which is a conclusion of Lemma 3 and Proposition 1.

**Corollary 1.** For matrices \(A, B\) and \(C\) in Proposition 1 we have

(a) \(\eta(F_C)i = 2\eta(F_A)i, 1 \leq i \leq 2^n, \eta(F_C)i = 0, 2^n < i \leq 2^n + 1,\)

(b) \(\eta(F_B)i = 2\eta(F_A)i, 1 \leq i \leq 2^n, \eta(F_B)i = 0, 2^n < i \leq 2^n + 1, i \neq T_{I_A}^A, i \neq 2T_{I_A}^A.\)

If \(e \in O_{F_A}(I_A)\) then

(c) \(\eta(F_B)_{2T_{I_A}^B} = 2(\eta(F_A)_{2T_{I_A}^B} - 1), \eta(F_B)_{2T_{I_A}^B} = 1 + \begin{cases} 2\eta(F_A)_{2T_{I_A}^B}, & 2T_{I_A}^B \leq 2^n, \\ 2T_{I_A}^B > 2^n. \end{cases}\)

If \(e \notin O_{F_A}(I_A)\) then

(d) \(\eta(F_B)_{2T_{I_A}^B} = 2(\eta(F_A)_{2T_{I_A}^B} - 2) + 2, \eta(F_B)_{2T_{I_A}^B} = 1 + \begin{cases} 2\eta(F_A)_{2T_{I_A}^B}, & 2T_{I_A}^B \leq 2^n, \\ 2T_{I_A}^B > 2^n. \end{cases}\)

Since \(\eta(F_B)_{2T_{I_A}^B}\) is odd, the neural networks \(B\) and \(C\) are not equivalent.

**Proof.** Observe that since \(2(\eta(F_A)_{2T_{I_A}^B} - 1) = 2(\eta(F_A)_{2T_{I_A}^B} - 2) + 2\) we could join (c) and (d). For the sake of clarity, we prefer this form.

From Proposition 1 we know that from each cycle \(O_{F_A}(x)\) we can obtain two cycles \(O_{F_A}(x, -1)\) and \(O_{F_A}(x, 1)\) with the same period and thus the cycle number of a given size of \(F_A\) is doubled in \(F_C\). This same argument is also true for \(F_B\) when the cycle \(O_{F_B}\) contains neither \(I_A\) nor \(-I_A\). When \(I_A\) or \(-I_A\) belongs to \(O_{F_A}(x)\) we know that if \(e \in O_{F_A}(I_A)\) then \(O_{F_A}(I_A) = O_{F_A}(-I_A)\) and the cycle \(O_{F_A}(I_A)\), which is of size \(T_{I_A}^F\), is transformed in the cycle \(O_{F_B}(\mu I_A, \mu)\) of size \(2T_{I_A}^F\). This is described by (c). If \(e \notin O_{F_A}(I_A)\) then \(O_{F_A}(I_A) \neq O_{F_A}(-I_A)\) and both are transformed in the cycle \(O_{F_B}(I_A, -1)\) of size
2T_{T_A}^p$, the cycle $O_{F_A}(I_A, 1)$ of size $T_{T_A}^p$ and the cycle $O_{F_A}(-I_A, -1)$ of size $T_{T_A}^p$. The last observation is trivial from the definition of $\eta$. □

**Proposition 2.** Let $\{A^i\}_{i=1}^I$ be a family of nonequivalent reverberation neural networks in $M_*(\mathbb{R})$. Then $\{B^i, C^i\}_{i=1}^I$ is a family of nonequivalent reverberation neural networks in $M_{*+1}(\mathbb{R})$, where $B^i$ and $C^i$ are built from $A^i$ in Theorem 1.

**Proof.** Suppose that there exist two equivalent neural networks in $\{B^i, C^i\}_{i=1}^I$. Then, it is sufficient to analyze the following cases:

(a) $\eta(F_{B^i}) = \eta(F_{B^j})$. Then we have that $\forall 1 \leq k \leq 2^n$, $k \neq T_{T_A}^p$ and $k \neq 2T_{T_A}^p$

$$\eta(F_{B^i}) = \eta(F_{B^j}) \Rightarrow \eta(F_{A^i}) = \eta(F_{A^j})$$

and from (c) and (d) in Corollary 1 one obtains $\eta(F_{A^i}) = \eta(F_{A^j})$.

(b) $\eta(F_{C^i}) = \eta(F_{C^j})$. Applying the same arguments as in (a) we conclude that $\eta(F_{A^i}) = \eta(F_{A^j})$, so (a) and (b) are in contradiction with the nonequivalence of $A_i$ and $A_j$.

(c) $\eta(F_{B^i}) = \eta(F_{C^i})$. Then

$$\eta(F_{B^i}) = 1 \begin{cases} \text{even}, & 2T_{T_A}^p \leq 2^n \\ 0, & 2T_{T_A}^p > 2^n \end{cases}$$

and from (c) and (d) in Corollary 1 one obtains $\eta(F_{A^i}) = \eta(F_{A^j})$.

Theorem 2. For any $n \in \mathbb{N}$ there exist $2^n$ nonequivalent reverberation neural networks in $M_*(\mathbb{R})$.

**Proof.** We proceed by induction on $n$. For $n=2$ the matrices $A^i$: $i=1,2,3,4$ given by

$$A^1 = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}, \quad A^2 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{pmatrix}, \quad A^3 = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}, \quad A^4 = \begin{pmatrix} \frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{pmatrix}$$

are in $M_2^*$, and have the following characteristics:

$$\eta(A^1) = (4, 0, 0, 0), \quad \eta(A^2) = (2, 1, 0, 0), \quad \eta(A^3) = (0, 2, 0, 0), \quad \eta(A^4) = (0, 0, 0, 1)$$

and then they are not equivalent. Since there exist $2^n$ nonequivalent neural networks for matrices belonging to $M_*(\mathbb{R})$ we can apply Proposition 2 in order to obtain $2^{n+1}$ nonequivalent neural networks belonging to $M_{*+1}^*$. □

By using Corollary 1 we get the following result which is given in [5].

**Corollary 2.** $\forall n \in \mathbb{N}$ there exists $A \in M_*(\mathbb{R})$ whose characteristic is given by

$$\eta(F_{A^i}) = 0 \text{ for } i \neq 2^n \text{ and } \eta(F_{A^2}) = \eta(F_{A^3}) = \eta(F_{A^4}) = 1.$$
Proof. Taking $n=2$ we have that $A^4$ given by Theorem 2 belongs to $M_2^*$ and its characteristic is $(0,0,0,1)$. Accepting that there exists $A \in M_n^*(\mathbb{R})$ with $\eta(A) = (0,\ldots,1)$, then by Corollary 1 we obtain $B \in M_{n+1}^*(\mathbb{R})$ with $\eta(B) = (0,\ldots,1)$ because $e_n \in O_{F_A}(I_A)$. □

4. Conclusion

The results shown in this work allow us to obtain a wide variety of nonequivalent dynamics when we consider the family of reverberation neural networks in $M_n^*(\mathbb{R})$. This kind of constructions can be applied for information storage where the information is codified in the cycles of the neural networks.

We desire to extend our construction to any function in $M_n^*(\mathbb{R})$. In this case Theorem 1 is true and we can build recursively neural networks in $M_n^*(\mathbb{R})$. Moreover, we can obtain an analogous result to Proposition 1 which allows us to know the behavior of neural networks of size $n+1$ in terms of those of size $n$. But, the characterization given in Lemma 3 for the equivalence of two functions in $M_n^*(\mathbb{R})$ is no longer true. For that, it is interesting to find an invariant in the general case.

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