Computational Aspects of Symbolic Dynamics

Part III: Turing Degrees

E. Jeandel

LORIA (Nancy, France)

Introduction

- The Turing degree of a point x expresses how hard it is to compute x
- Turing reducibility is a partial preorder : $x \leq_T y$ if x is easier to compute than y
- Easiest points for the Turing reducibility are *computable* points, i.e. points where there is an algorithm that computes x_i on input i.

Definitions

x and y are assumed to be in a Cantor Space, as in Lecture 1.

Definition

$x \leq_T y$ if there is a (type 2) algorithm that on input y computes x.

"If we are given y for free, x is easy to compute".
Clearly a preorder :

- $x \leq_T x$.
- If $x \leq_T y$ and $y \leq_T z$, we can compute x from z : To obtain x_i given z, simulate the algorithm that computes x_i given y. Every time it asks about the value of y_j for some j, simulate the algorithm that computes y_j from z.

Some properties of the order

- Minimal elements : x is computable iff $x \leq_T y$ for all y
- No maximal element : Given y, $\{x | x \leq_T y\}$ is always countable.
- Upper semi-lattice : Given $x, y \in A^{\mathbb{N}}$, define z by

$$\begin{cases} z_{2i} & = & x_i \\ z_{2i+1} & = & y_i \end{cases}$$

Then $z = x \oplus y$ is the lowest upper bound of (x, y).

Turing degrees

Definition

$x \equiv_T y$ is $x \leq_T y$ and $y \leq_T x$ (x and y are as hard to compute)
The equivalence classes for $\equiv_T$ are called Turing degrees. We denote by $\deg_T x$ the equivalence class of x

The equivalence class of computable points is usually denoted $\emptyset$.

Turing degrees and subshifts

- If f is a computable function, then $f(x) \leq_T x$ (clear)
- If f and f^{-1} are computable, $f(x) \equiv_T x$.

Corollary

$\deg_T S = \{\deg_T x, x \in S\}$ *is a conjugacy invariant*

What can we say about $\deg_T S$?

Point of reference

Recall that effective subshifts are examples of effectively closed sets.

- A lot of literature on effectively closed sets, and their sets of Turing degrees.
- Can we obtain the same sets of Turing degrees with effective subshifts and with effectively closed sets ?

First Result

Theorem (essentially Myers [Mye74], see also [CDK08])

There exists an effective 1D subshift with no computable points

As a corollary, by Aubrun-Sablik [AS], there exists a 2D SFT with no computable points. (Why ?)

A first lemma

Lemma (Miller [Mil12])

Let $S \subseteq \{0,1\}^+$ be a set of nonempty words and $c > 1/2$ so that

$$\sum_{w \in S} c^{|w|} \leq 2c - 1$$

Then the subshift over $\{0,1\}^{\mathbb{N}}$ avoiding all of S is nonempty.

In particular if S contains exactly one word of each size n for $n \geq 5$, the condition is satisfied (take $c = \frac{\sqrt{5}-1}{2}$)
Proof is short, refer to [Mil12].

Proof
Following [CDK08]

- Let f_i be an enumeration of all algorithms
- x is computable if there exists i so that $x_j = f_i(j)$

Let $u_i = f_i(0)f_i(1)\dots f_i(i+5)$ (if f_i halts on inputs $0\dots i+5$)

- u_i is of length $i+6$.
- By the previous lemma, there exists a (effective) subshift that forbids all u_i.
- This subshift contains no computable point.

Subshifts with no computable points

It turns out we can say a bit more about $\deg_S$.

Theorem (J.-Vanier [JV12])

Let S be any nonempty subshift.
Then either S contains a computable point or there exists a degree d so that S contains points of any degree above d.

If S does not contain a computable point, it must contain arbitrarily complex points.

Proof (in 1D)

- W.l.o.g we may suppose that S is a minimal subshift (S does not contain a proper nonempty subshift).
- All points in a minimal subshift are uniformly recurrent : For every finite word w, there exists a size n so that w appear in all windows of size n.
- This minimal subshift cannot contain periodic points (periodic points are computable)

It is easy to see that such a subshift must be of cardinality $2^{\aleph_0}$ (the continuum). To prove the theorem we will provide an effective embedding of $2^{\aleph_0}$ to S

Proof (in 1D)

We start from an infinite word $u \in S$ and a word $x \in \{0,1\}^{\mathbb{N}}$ and will build a word $f(u,x)$ so that

- $f(u,x) \in S$
- $f(u,x)$ is computable given x and u
- x can be recovered given $f(x,u)$.

So if we take x so that $\deg x \geq \deg u$, then

- $\deg f(u,x) \leq \deg x$ (u can be computed given x and $f(u,x)$ can be computed given x and u)
- $\deg x \leq \deg f(u,x)$ (x can be computed given $f(x,u)$)

So $\deg f(u,x) = \deg x$.

Proof (in 1D)

We start from an infinite word $u \in S$ and a word $x \in \{0,1\}^{\mathbb{N}}$.
Suppose to simplify u is over the binary alphabet $\{a, b\}$.
We build inductively words w_i so that $f(u,x) = \lim w_i$.

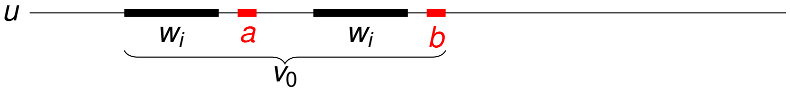
- $w_0 = a$
- Suppose w_i is defined.
- w_i appears infinitely many times in u (remember u is uniformly recurrent)
- Look at two consecutive occurences, and where they differ (they must differ, otherwise u is periodic)

  u ———————————————————————
- If $x_{i+1} = 0$, let $w_{i+1} = v_0$ otherwise $w_{i+1} = v_1$.

Proof (in 1D)

We start from an infinite word $u \in S$ and a word $x \in \{0,1\}^{\mathbb{N}}$.
Suppose to simplify u is over the binary alphabet $\{a,b\}$.
We build inductively words w_i so that $f(u,x) = \lim w_i$.

- $w_0 = a$
- Suppose w_i is defined.
- w_i appears infinitely many times in u (remember u is uniformly recurrent)
- Look at two consecutive occurences, and where they differ (they must differ, otherwise u is periodic)

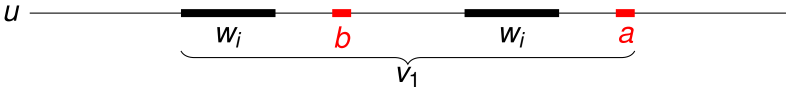


- If $x_{i+1} = 0$, let $w_{i+1} = v_0$ otherwise $w_{i+1} = v_1$.

Proof (in 1D)

We start from an infinite word $u \in S$ and a word $x \in \{0,1\}^{\mathbb{N}}$.
Suppose to simplify u is over the binary alphabet $\{a, b\}$.
We build inductively words w_i so that $f(u, x) = \lim w_i$.

- $w_0 = a$
- Suppose w_i is defined.
- w_i appears infinitely many times in u (remember u is uniformly recurrent)
- Look at two consecutive occurences, and where they differ (they must differ, otherwise u is periodic)

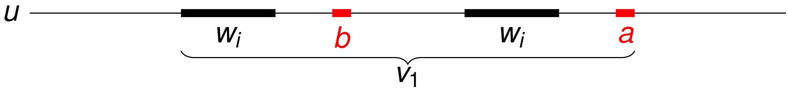


- If $x_{i+1} = 0$, let $w_{i+1} = v_0$ otherwise $w_{i+1} = v_1$.

Proof (in 1D)

We start from an infinite word $u \in S$ and a word $x \in \{0,1\}^{\mathbb{N}}$.
Suppose to simplify u is over the binary alphabet $\{a, b\}$.
We build inductively words w_i so that $f(u, x) = \lim w_i$.

- $w_0 = a$
- Suppose w_i is defined.
- w_i appears infinitely many times in u (remember u is uniformly recurrent)
- Look at two consecutive occurences, and where they differ (they must differ, otherwise u is periodic)



- If $x_{i+1} = 0$, let $w_{i+1} = v_0$ otherwise $w_{i+1} = v_1$.

What it says

- If a subshift contains no computable points, it must contain arbitrarily complex points.
- It turns out that there are effectively closed sets where neither are true
- There are some sets of Turing degrees that can be achieved by effectively closed sets but not by subshifts.

We still do not know what sets of Turing degrees can be achieved by subshifts.

- Positive Result : Every set of Turing degrees than can be achieved by effectively closed sets AND that contains a computable point can be achieved by an effective subshift (and a 2D SFT). [CDTW12, JV12].

A solution : Muchnik equivalence

- View a set S as a set of possible solutions to a given problem
- $S \leq_w S'$ if it is easier to display a solution to S than a solution to S'
- Knowing some solution to S is enough to obtain a solution to S'.

(w in $\leq_w$ is for "weak". There is a notion of a strong (Medvedev) reduction.)

Muchnik reduction

Definitions

Definition

$S \leq_w S'$ if for every $y \in S'$, there exists $x \in S$ so that $x \leq_T y$.
S and S' are Muchnik equivalent if $S \leq_w S'$ and $S' \leq_w S$.
The Muchnik degree of a set is its equivalence class for $\equiv_w$.

Muchnik equivalence means somehow that S and S' have the same “minimal” elements.

The Muchnik degree of a subshift is also a conjugacy invariant.

Examples

- If $A \subseteq B$, $B \leq_w A$.
- For $A \subseteq \{0,1\}^{\mathbb{N}}$, $B = A \otimes \{a,b\}^{\mathbb{N}} \subseteq \{(0,a),(1,a),(0,b),(1,b)\}^{\mathbb{N}}$
  - $A \leq B$. Given $x \in B$, just forget about the a and b's to obtain a point in A.
  - $B \leq A$. Given $y \in A$, add the symbol a everywhere to obtain a point in B.
- $X_{\mathcal{F}}$ biinfinite words that forbid $\mathcal{F}$, and $X^+_{\mathcal{F}}$ infinite words that forbid $\mathcal{F}$.
  - $X^+_{\mathcal{F}} \leq X_{\mathcal{F}}$ is clear
  - $X_{\mathcal{F}} \leq X^+_{\mathcal{F}}$? ? ? ? Not always.

Main theorem

Theorem (Miller [Mil12])

For every effectively closed set $A \subseteq \{0,1\}^{\mathbb{N}}$, there is an effective subshift $S \subseteq \{0,1,2,3\}^{\mathbb{N}}$ so that A and S are Muchnik equivalent.

So the Muchnik degrees of subshifts are the same as the Muchnik degrees of effectively closed sets.

A digression about the Thue-Morse sequence

Let S_1 be the subshift over $\{a, b\}$ that forbids

$$\{xyxyx, x \in \{a, b\}^+, y \in \{a, b\}^\star\}$$

This subshift is nonempty. It indeed contains the Thue-Morse sequence u

$$u = \lim t^n(a)$$

where

$$t : a \to ab, b \to ba$$

$$abbabaabbaababbabaababbaabbabaab\ldots$$

This subshift S_1 is uniformly recurrent : All words of S_1 contain the same factors as the Thue-Morse sequence.

More on Thue-Morse

Let's look at the words $u_n = t^n(aa)$.

- We know exactly where the words u_n are in the Thue-Morse sequence.
  - u_n occur exactly at positions $2^n m$ where m is such that m and $m+1$ have an even number of 1 in their binary representation.
- u_n is not a prefix of u_m if $n \neq m$.
  - If $t^n(aa)$ is a prefix of $t^m(aa)$ then aa is a prefix of $t^{m-n}(aa)$. But $t^{m-n}(aa)$ begins with $ab\ldots$

The construction

We start from a set $A \subseteq \{0,1\}^{\mathbb{N}}$, and we will "embed" it into a subshift $S \subseteq (\{a,b\} \times \{0,1\})^{\mathbb{N}}$

- For each word $x \in A$ and every word w in S_1, we embed x in w by putting the letter x_n above every position in w where the word $t^n(aa)$ appears. The other positions are arbitrary. Let $f(x,w)$ denote the new word.
- The construction is well defined because $t^n(aa)$ and $t^m(aa)$ cannot appear at the same position if $n \neq m$.

The construction

$x = 1011110110101..$

$b\ a\ a\ b\ b\ a\ a\ b\ a\ b\ b\ a\ b\ a\ a\ b\ a\ b\ b\ a\ a\ b\ b\ a\ b\ a\ a\ b\ \dots$

The construction

$x = 1011110110101..$

$b\ a\ a\ b\ b\ a\ a\ b\ a\ b\ b\ a\ b\ a\ a\ b\ a\ b\ b\ a\ a\ b\ b\ a\ b\ a\ a\ b\ \ldots$

$t^0(aa) \quad = \quad \text{aa}$

The construction

$x = 1011110110101..$

$$
\begin{array}{cccccccccccccccccccccccccccc}
 & 1 & & & & 1 & & & & & & & & 1 & & & & & & 1 & & & & & & 1 & & \\
b & a & a & b & b & a & a & b & a & b & b & a & b & a & a & b & a & b & b & a & a & b & b & a & b & a & a & b \ \ldots
\end{array}
$$

$t^0(aa) \quad = \quad \text{aa}$

The construction

$x = 1011110110101..$

	1				1								1						1						1			
b	a	a	b	b	a	a	b	a	b	b	a	b	a	a	b	a	b	b	a	a	b	b	a	b	a	a	b	$\ldots$

$$t^0(aa) = \text{aa}$$
$$t^1(aa) = \text{abab}$$

The construction

$x = 1011110110101..$

	1				1	0							1	0					1						1			
b	a	a	b	b	a	a	b	a	b	b	a	b	a	a	b	a	b	b	a	a	b	b	a	b	a	a	b	$\dots$

$$\begin{aligned} t^0(aa) &= \text{aa} \\ t^1(aa) &= \text{abab} \end{aligned}$$

The construction

$x = 101110110101..$

	1				1	0							1	0					1						1			
b	a	a	b	b	a	a	b	a	b	b	a	b	a	a	b	a	b	b	a	a	b	b	a	b	a	a	b	$\dots$

$$
\begin{aligned}
t^0(aa) &= \text{aa} \\
t^1(aa) &= \text{abab} \\
t^2(aa) &= \text{abbaabba}
\end{aligned}
$$

The construction

$x = 101110110101..$

	1				1	0							1	0		1			1						1			
b	a	a	b	b	a	a	b	a	b	b	a	b	a	a	b	a	b	b	a	a	b	b	a	b	a	a	b	$\dots$

$$t^0(aa) = \text{aa}$$
$$t^1(aa) = \text{abab}$$
$$t^2(aa) = \text{abbaabba}$$

The construction

$x = 101110110101..$

	1				1	0							1	0		1			1						1			
b	*a*	*a*	*b*	*b*	*a*	*a*	*b*	*a*	*b*	*b*	*a*	*b*	*a*	*a*	*b*	*a*	*b*	*b*	*a*	*a*	*b*	*b*	*a*	*b*	*a*	*a*	*b*	...

$$\begin{aligned} t^0(aa) &= \text{aa} \\ t^1(aa) &= \text{abab} \\ t^2(aa) &= \text{abbaabba} \\ \cdots & \quad \ldots \end{aligned}$$

The construction

$x = 101110110101..$

$$\begin{array}{ccccccccccccccccccccccccccccc} \star & 1 & \star & \star & \star & 1 & 0 & \star & \star & \star & \star & \star & \star & 1 & 0 & \star & 1 & \star & \star & 1 & \star & \star & \star & \star & \star & 1 & \star & \star & \\ b & a & a & b & b & a & a & b & a & b & b & a & b & a & a & b & a & b & b & a & a & b & b & a & b & a & a & b & \dots \end{array}$$

$$\begin{array}{rcl} t^0(aa) & = & \text{aa} \\ t^1(aa) & = & \text{abab} \\ t^2(aa) & = & \text{abbaabba} \\ \dots & & \dots \end{array}$$

The construction

S is the set of all words we obtain this way, starting from any $x \in A$ and $w \in S_1$.

- S is an effective subshift (it can be defined by forbidden patterns)
- $S \leq_w A$. If we are given $x \in A$, we can build a point in S by considering the previous construction for u the Thue-Morse word (recall we know where each pattern $t^n(aa)$ appear in the TM-word), putting 0 everywhere else.
- $A \leq_w S$. Given a point of S, define x_n to be the letter above the first occurence of $t^n(aa)$. Then $x \in A$.

The result (again)

Theorem (Miller [Mil12])

For every effectively closed set $A \subseteq \{0,1\}^{\mathbb{N}}$, there is an effective subshift $S \subseteq \{0,1,2,3\}^{\mathbb{Z}}$ so that A and S are Muchnik equivalent.

Corollary (Simpson [Sim11])

For every effectively closed set $A \subseteq \{0,1\}^{\mathbb{N}}$, there is a 2D SFT $S \subseteq \{0,1,2,3\}^{\mathbb{Z}^{\nvDash}}$ so that A and S are Muchnik equivalent.

Again, by a careful examination of Aubrun-Sablik (Why ?)

Bibliography I

- Nathalie Aubrun and Mathieu Sablik, *Simulation of effective subshifts by two-dimensional SFT and a generalization*, preprint.
- Douglas Cenzer, Ali Dashti, and Jonathan L. F. King, *Computable symbolic dynamics*, Mathematical Logic Quarterly **54** (2008), no. 5, 460–469.
- Douglas Cenzer, Ali Dashti, Ferit Toska, and Sebastian Wyman, *Computability of Countable Subshifts in One Dimension*, Theory of Computing Systems (2012).
- Emmanuel Jeandel and Pascal Vanier, *Turing degrees of multidimensional SFTs*, Theoretical Computer Science (2012).
- Joseph S. Miller, *Two Notes on subshifts*, Proceedings of the American Mathematical Society **140** (2012), no. 5, 1617–1622.

Bibliography II

- Dale Myers, *Non Recursive Tilings of the Plane II*, Journal of Symbolic Logic **39** (1974), no. 2, 286–294.
- Stephen G. Simpson, *Medvedev Degrees of 2-Dimensional Subshifts of Finite Type*, Ergodic Theory and Dynamical Systems (2011).