Entropy and mixing for $\mathbb{Z}^d$ SFTs

Ronnie Pavlov

University of Denver
www.math.du.edu/∼rpavlov

1st School on Dynamical Systems and Computation (DySyCo)
CMM, Santiago, Chile

Definitions

- Begin with finite set A called **alphabet**; elements called **letters**

Definitions

- Begin with finite set A called **alphabet**; elements called **letters**
- A **pattern** is a member of A^S for some finite $S \subseteq \mathbb{Z}^d$, called the **shape**

Definitions

- Begin with finite set A called **alphabet**; elements called **letters**
- A **pattern** is a member of A^S for some finite $S \subseteq \mathbb{Z}^d$, called the **shape**
- Any set $\mathcal{F}$ of patterns defines a **subshift** $X = X(\mathcal{F}) := \{\omega \in A^{\mathbb{Z}^d} \ : \ \omega \text{ does not contain any pattern from } \mathcal{F}\}$

Definitions

- Begin with finite set A called **alphabet**; elements called **letters**
- A **pattern** is a member of A^S for some finite $S \subseteq \mathbb{Z}^d$, called the **shape**
- Any set $\mathcal{F}$ of patterns defines a **subshift** $X = X(\mathcal{F}) := \{\omega \in A^{\mathbb{Z}^d} \ : \ \omega \text{ does not contain any pattern from } \mathcal{F}\}$
- Equivalently, a **subshift** is a closed shift-invariant subset X of $A^{\mathbb{Z}^d}$ w.r.t. the product topology

Definitions

- Begin with finite set A called **alphabet**; elements called **letters**
- A **pattern** is a member of A^S for some finite $S \subseteq \mathbb{Z}^d$, called the **shape**
- Any set $\mathcal{F}$ of patterns defines a **subshift** $X = X(\mathcal{F}) := \{\omega \in A^{\mathbb{Z}^d} \ : \ \omega \text{ does not contain any pattern from } \mathcal{F}\}$
- Equivalently, a **subshift** is a closed shift-invariant subset X of $A^{\mathbb{Z}^d}$ w.r.t. the product topology
- Subshifts are topological dynamical systems (t.d.s.) when endowed with $\mathbb{Z}^d$-action by shifts

Definitions

- Important problem: study maps between topological dynamical systems

Definitions

- Important problem: study maps between topological dynamical systems
- Any continuous shift-commuting map between subshifts has a simple structure

Definitions

- Important problem: study maps between topological dynamical systems
- Any continuous shift-commuting map between subshifts has a simple structure
- Any such map ϕ is a **sliding block code**; exists n so that $(\phi(x))(0)$ is determined by $x(\{-n, \ldots, n\}^d)$

Definitions

- Important problem: study maps between topological dynamical systems
- Any continuous shift-commuting map between subshifts has a simple structure
- Any such map ϕ is a **sliding block code**; exists n so that $(\phi(x))(0)$ is determined by $x(\{-n, \ldots, n\}^d)$
- Say X **factors onto** Y if there is a surjective SBC from X to Y

Definitions

- Important problem: study maps between topological dynamical systems
- Any continuous shift-commuting map between subshifts has a simple structure
- Any such map ϕ is a **sliding block code**; exists n so that $(\phi(x))(0)$ is determined by $x(\{-n, \ldots, n\}^d)$
- Say X **factors onto** Y if there is a surjective SBC from X to Y
- Say X **embeds into** Y if there is an injective SBC from X to Y

Definitions

- Important problem: study maps between topological dynamical systems
- Any continuous shift-commuting map between subshifts has a simple structure
- Any such map ϕ is a **sliding block code**; exists n so that $(\phi(x))(0)$ is determined by $x(\{-n, \ldots, n\}^d)$
- Say X **factors onto** Y if there is a surjective SBC from X to Y
- Say X **embeds into** Y if there is an injective SBC from X to Y
- Say X is **conjugate to** Y if there is a bijective SBC from X to Y

Definitions

- When $\mathcal{F}$ is finite, we say $X(\mathcal{F})$ is a **shift of finite type** or **SFT**

Definitions

- When $\mathcal{F}$ is finite, we say $X(\mathcal{F})$ is a **shift of finite type** or **SFT**
- Special case: if $\mathcal{F}$ consists only of pairs of adjacent letters, X is a **nearest neighbor SFT**

Definitions

- When $\mathcal{F}$ is finite, we say $X(\mathcal{F})$ is a **shift of finite type** or **SFT**
- Special case: if $\mathcal{F}$ consists only of pairs of adjacent letters, X is a **nearest neighbor SFT**
- Every SFT is conjugate to a nearest-neighbor SFT, so w.l.o.g. can reduce to this case

Definitions

- When $\mathcal{F}$ is finite, we say $X(\mathcal{F})$ is a **shift of finite type** or **SFT**
- Special case: if $\mathcal{F}$ consists only of pairs of adjacent letters, X is a **nearest neighbor SFT**
- Every SFT is conjugate to a nearest-neighbor SFT, so w.l.o.g. can reduce to this case
- Wish to study conjugacies/factors between SFTs

Definitions

- When $\mathcal{F}$ is finite, we say $X(\mathcal{F})$ is a **shift of finite type** or **SFT**
- Special case: if $\mathcal{F}$ consists only of pairs of adjacent letters, X is a **nearest neighbor SFT**
- Every SFT is conjugate to a nearest-neighbor SFT, so w.l.o.g. can reduce to this case
- Wish to study conjugacies/factors between SFTs
- Two useful tools: entropy and mixing

Definitions

- The **language** $L(X)$ of an SFT X is the set of patterns appearing in points of X

Definitions

- The **language** $L(X)$ of an SFT X is the set of patterns appearing in points of X
- **Topological entropy** of X is $h(X) := \lim_{n\to\infty} \frac{\log |L(X) \cap A^{\{1,\dots,n\}^d}|}{n^d}$

Definitions

- The **language** $L(X)$ of an SFT X is the set of patterns appearing in points of X
- **Topological entropy** of X is $h(X) := \lim_{n\to\infty} \frac{\log |L(X) \cap A^{\{1,\ldots,n\}^d}|}{n^d}$
- Measures "complexity" of X

Definitions

- The **language** $L(X)$ of an SFT X is the set of patterns appearing in points of X
- **Topological entropy** of X is $h(X) := \lim_{n\to\infty} \frac{\log |L(X) \cap A^{\{1,\dots,n\}^d}|}{n^d}$
- Measures "complexity" of X
    - If X factors onto Y, then $h(X) \geq h(Y)$ (PROVE)

Definitions

- The **language** $L(X)$ of an SFT X is the set of patterns appearing in points of X
- **Topological entropy** of X is $h(X) := \lim_{n\to\infty} \frac{\log |L(X) \cap A^{\{1,\ldots,n\}^d}|}{n^d}$
- Measures "complexity" of X
  - If X factors onto Y, then $h(X) \geq h(Y)$ (PROVE)
  - Topological entropy is a conjugacy invariant

Definitions

- For any $w \in L(X)$, define **cylinder set** $[w] = \{x \in X \ : \ x(S) = w\}$

Definitions

- For any $w \in L(X)$, define **cylinder set** $[w] = \{x \in X \; : \; x(S) = w\}$
  - Cylinder sets are clopen sets generating topology of X

Definitions

- For any $w \in L(X)$, define **cylinder set** $[w] = \{x \in X \ : \ x(S) = w\}$
  - Cylinder sets are clopen sets generating topology of X
- For $d = 1$, X is **topologically mixing** if $\forall v, w \in L(X), \exists N$ s.t. $\forall n > N$, $[v] \cap \sigma^{-n}[w] \neq \varnothing$

Definitions

- For any $w \in L(X)$, define **cylinder set** $[w] = \{x \in X \ : \ x(S) = w\}$
  - Cylinder sets are clopen sets generating topology of X
- For $d = 1$, X is **topologically mixing** if $\forall v, w \in L(X), \exists N$ s.t. $\forall n > N$, $[v] \cap \sigma^{-n}[w] \neq \varnothing$
  - Topological mixing is conjugacy invariant

Definitions

- For any $w \in L(X)$, define **cylinder set** $[w] = \{x \in X \ : \ x(S) = w\}$
  - Cylinder sets are clopen sets generating topology of X
- For $d = 1$, X is **topologically mixing** if $\forall v, w \in L(X), \exists N$ s.t. $\forall n > N$, $[v] \cap \sigma^{-n}[w] \neq \varnothing$
  - Topological mixing is conjugacy invariant
  - $\mathbb{Z}$ SFTs can be "decomposed" into mixing pieces

Definitions

- For any $w \in L(X)$, define **cylinder set** $[w] = \{x \in X \ : \ x(S) = w\}$
  - Cylinder sets are clopen sets generating topology of X
- For $d = 1$, X is **topologically mixing** if $\forall v, w \in L(X), \exists N$ s.t. $\forall n > N$, $[v] \cap \sigma^{-n}[w] \neq \varnothing$
  - Topological mixing is conjugacy invariant
  - $\mathbb{Z}$ SFTs can be "decomposed" into mixing pieces
  - Topological mixing implies positive entropy and dense periodic points (PROVE)

Factors/embeddings for $d = 1$

- In $d = 1$, entropy is main tool controlling factor structure

Factors/embeddings for $d = 1$

- In $d = 1$, entropy is main tool controlling factor structure
- **Theorem:** (Boyle/Marcus) If $h(X) > h(Y)$ and for every periodic point in X there is a point in Y with the same period, then X factors onto Y

Factors/embeddings for $d = 1$

- In $d = 1$, entropy is main tool controlling factor structure
- **Theorem:** (Boyle/Marcus) If $h(X) > h(Y)$ and for every periodic point in X there is a point in Y with the same period, then X factors onto Y
- **Theorem:** (Krieger) If $h(X) < h(Y)$ and for every n, there are at least as many points with period n in Y as there are in X, then X embeds into Y

Factors/embeddings for $d = 1$

- In $d = 1$, entropy is main tool controlling factor structure
- **Theorem:** (Boyle/Marcus) If $h(X) > h(Y)$ and for every periodic point in X there is a point in Y with the same period, then X factors onto Y
- **Theorem:** (Krieger) If $h(X) < h(Y)$ and for every n, there are at least as many points with period n in Y as there are in X, then X embeds into Y
- Can prove by reducing to case where X and Y are topologically mixing

Mixing for $d > 1$

- Mixing much more complicated for $d > 1$

Mixing for $d > 1$

- Mixing much more complicated for $d > 1$
- X is **topologically mixing** if $\forall v, w \in L(X)$, $\exists N$ s.t. $[v] \cap \sigma^t[w] \neq \varnothing$ for all $t \in \mathbb{Z}^d$ with $\|t\| > N$

Mixing for $d > 1$

- Mixing much more complicated for $d > 1$
- X is **topologically mixing** if $\forall v, w \in L(X)$, $\exists N$ s.t. $[v] \cap \sigma^t[w] \neq \varnothing$ for all $t \in \mathbb{Z}^d$ with $\|t\| > N$
- Uniform mixing: require N to be independent of v, w

Mixing for $d > 1$

- Mixing much more complicated for $d > 1$
- X is **topologically mixing** if $\forall v, w \in L(X)$, $\exists N$ s.t. $[v] \cap \sigma^t[w] \neq \varnothing$ for all $t \in \mathbb{Z}^d$ with $\|t\| > N$
- Uniform mixing: require N to be independent of v, w
- X is **strongly irreducible** if $\exists N$ s.t. $\forall v, w \in L(X)$, $[v] \cap \sigma^t[w] \neq \varnothing$ for all $t \in \mathbb{Z}^d$ with $\|t\| > N$

Mixing for $d > 1$

- Mixing much more complicated for $d > 1$
- X is **topologically mixing** if $\forall v, w \in L(X)$, $\exists N$ s.t. $[v] \cap \sigma^t[w] \neq \varnothing$ for all $t \in \mathbb{Z}^d$ with $\|t\| > N$
- Uniform mixing: require N to be independent of v, w
- X is **strongly irreducible** if $\exists N$ s.t. $\forall v, w \in L(X)$, $[v] \cap \sigma^t[w] \neq \varnothing$ for all $t \in \mathbb{Z}^d$ with $\|t\| > N$
- More subtleties: additional geometry in $\mathbb{Z}^d$ allows for other interpretations

Mixing for $d > 1$

- Mixing much more complicated for $d > 1$
- X is **topologically mixing** if $\forall v, w \in L(X)$, $\exists N$ s.t. $[v] \cap \sigma^t[w] \neq \varnothing$ for all $t \in \mathbb{Z}^d$ with $\|t\| > N$
- Uniform mixing: require N to be independent of v, w
- X is **strongly irreducible** if $\exists N$ s.t. $\forall v, w \in L(X)$, $[v] \cap \sigma^t[w] \neq \varnothing$ for all $t \in \mathbb{Z}^d$ with $\|t\| > N$
- More subtleties: additional geometry in $\mathbb{Z}^d$ allows for other interpretations
- X is **block gluing** if $\exists N$ s.t. $\forall v, w \in L(X)$ with shapes rectangular prisms, $[v] \cap \sigma^t[w] \neq \varnothing$ for all $t \in \mathbb{Z}^d$ with $\|t\| > N$

Mixing for $d > 1$

- Mixing much more complicated for $d > 1$
- X is **topologically mixing** if $\forall v, w \in L(X)$, $\exists N$ s.t. $[v] \cap \sigma^t[w] \neq \varnothing$ for all $t \in \mathbb{Z}^d$ with $\|t\| > N$
- Uniform mixing: require N to be independent of v, w
- X is **strongly irreducible** if $\exists N$ s.t. $\forall v, w \in L(X)$, $[v] \cap \sigma^t[w] \neq \varnothing$ for all $t \in \mathbb{Z}^d$ with $\|t\| > N$
- More subtleties: additional geometry in $\mathbb{Z}^d$ allows for other interpretations
- X is **block gluing** if $\exists N$ s.t. $\forall v, w \in L(X)$ with shapes rectangular prisms, $[v] \cap \sigma^t[w] \neq \varnothing$ for all $t \in \mathbb{Z}^d$ with $\|t\| > N$
- When $d = 1$, all of these notions are identical

Mixing for $d > 1$

- The **domino tiling** $\mathbb{Z}^2$ SFT D is the set of all ways of tiling the plane with dominoes.

Mixing for $d > 1$

- The **domino tiling** $\mathbb{Z}^2$ SFT D is the set of all ways of tiling the plane with dominoes.
- Strictly speaking, this is a tiling system, but easily realizable as an SFT.

Mixing for $d > 1$

- The **domino tiling** $\mathbb{Z}^2$ SFT D is the set of all ways of tiling the plane with dominoes.
- Strictly speaking, this is a tiling system, but easily realizable as an SFT.
  - $A = \{T, B, L, R\}$ (Think of as the top, bottom, left, or right halves of a domino)

Mixing for $d > 1$

- The **domino tiling** $\mathbb{Z}^2$ SFT D is the set of all ways of tiling the plane with dominoes.
- Strictly speaking, this is a tiling system, but easily realizable as an SFT.
  - $A = \{T, B, L, R\}$ (Think of as the top, bottom, left, or right halves of a domino)
  - Halves of dominoes must appear together.

Mixing for $d > 1$

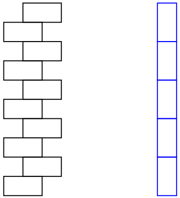
- The **domino tiling** $\mathbb{Z}^2$ SFT D is the set of all ways of tiling the plane with dominoes.

Mixing for $d > 1$

- The **domino tiling** $\mathbb{Z}^2$ SFT D is the set of all ways of tiling the plane with dominoes.
- D is topologically mixing because large legal rectangular patterns can coexist when they are far apart...

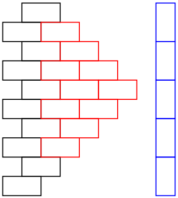
Mixing for $d > 1$

- The **domino tiling** $\mathbb{Z}^2$ SFT D is the set of all ways of tiling the plane with dominoes.
- D is topologically mixing because large legal rectangular patterns can coexist when they are far apart...



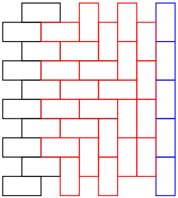
Mixing for $d > 1$

- The **domino tiling** $\mathbb{Z}^2$ SFT D is the set of all ways of tiling the plane with dominoes.
- D is topologically mixing because large legal rectangular patterns can coexist when they are far apart...



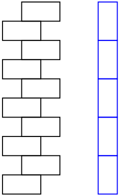
Mixing for $d > 1$

- The **domino tiling** $\mathbb{Z}^2$ SFT D is the set of all ways of tiling the plane with dominoes.
- D is topologically mixing because large legal rectangular patterns can coexist when they are far apart...



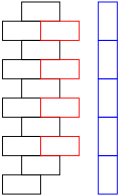
Mixing for $d > 1$

- The **domino tiling** $\mathbb{Z}^2$ SFT D is the set of all ways of tiling the plane with dominoes.
- D is topologically mixing because large legal rectangular patterns can coexist when they are far apart... but large patterns might need large distances, so it is not block gluing.



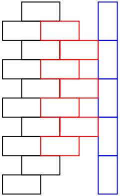
Mixing for $d > 1$

- The **domino tiling** $\mathbb{Z}^2$ SFT D is the set of all ways of tiling the plane with dominoes.
- D is topologically mixing because large legal rectangular patterns can coexist when they are far apart... but large patterns might need large distances, so it is not block gluing.



Mixing for $d > 1$

- The **domino tiling** $\mathbb{Z}^2$ SFT D is the set of all ways of tiling the plane with dominoes.
- D is topologically mixing because large legal rectangular patterns can coexist when they are far apart... but large patterns might need large distances, so it is not block gluing.



Mixing for $d > 1$

- The **southeast shift** S is the $\mathbb{Z}^2$ SFT with $A = \{0, 1\}$ and $\mathcal{F} = \left\{ \begin{smallmatrix} 00 \\ 01 \end{smallmatrix} \right\}$.

Mixing for $d > 1$

- The **southeast shift** S is the $\mathbb{Z}^2$ SFT with $A = \{0, 1\}$ and $\mathcal{F} = \{ \begin{smallmatrix} 00 \\ 01 \end{smallmatrix} \}$.
- S is block gluing because any two legal rectangular patterns with distance at least 2 can coexist...

Mixing for $d > 1$

- The **southeast shift** S is the $\mathbb{Z}^2$ SFT with $A = \{0, 1\}$ and $\mathcal{F} = \left\{ \begin{smallmatrix} 00 \\ 01 \end{smallmatrix} \right\}$.
- S is block gluing because any two legal rectangular patterns with distance at least 2 can coexist...

$$\begin{array}{|ccc|}\hline 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ \hline \end{array} \qquad \begin{array}{|cccc|}\hline 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline \end{array}$$

Mixing for $d > 1$

- The **southeast shift** S is the $\mathbb{Z}^2$ SFT with $A = \{0, 1\}$ and $\mathcal{F} = \left\{ \begin{smallmatrix} 00 \\ 01 \end{smallmatrix} \right\}$.
- S is block gluing because any two legal rectangular patterns with distance at least 2 can coexist...

1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	0	1	0	0	1
1	1	1	0	1	1	1	1	1	1	0	0	1
1	1	1	0	0	1	1	1	0	1	0	0	1
1	1	1	1	0	1	1	1	1	1	1	1	1
1	1	1	0	1	0	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1

Mixing for $d > 1$

- The **southeast shift** S is the $\mathbb{Z}^2$ SFT with $A = \{0, 1\}$ and $\mathcal{F} = \left\{ \begin{smallmatrix} 00 \\ 01 \end{smallmatrix} \right\}$.
- S is block gluing because any two legal rectangular patterns with distance at least 2 can coexist...

1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	0	1	0	0	1
1	1	1	0	1	1	1	1	1	1	0	0	1
1	1	1	0	0	1	1	1	0	1	0	0	1
1	1	1	1	0	1	1	1	1	1	1	1	1
1	1	1	0	1	0	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1

Mixing for $d > 1$

- The **southeast shift** S is the $\mathbb{Z}^2$ SFT with $A = \{0, 1\}$ and $\mathcal{F} = \{ \begin{smallmatrix} 00 \\ 01 \end{smallmatrix} \}$.
- S is block gluing because any two legal rectangular patterns with distance at least 2 can coexist...

1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	0	1	0	0	1
1	1	1	0	1	1	1	1	1	1	0	0	1
1	1	1	0	0	1	1	1	0	1	0	0	1
1	1	1	1	0	1	1	1	1	1	1	1	1
1	1	1	0	1	0	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1

Mixing for $d > 1$

- The **southeast shift** S is the $\mathbb{Z}^2$ SFT with $A = \{0, 1\}$ and $\mathcal{F} = \left\{ \begin{smallmatrix} 00 \\ 01 \end{smallmatrix} \right\}$.
- S is block gluing because any two legal rectangular patterns with distance at least 2 can coexist... but the same is not true of non-rectangular patterns, so S is not strongly irreducible.

$$\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & & & & & & & & \\
0 & & & & & & & & \\
0 & & & & 1 & & & & \\
0 & & & & & & & & \\
0 & & & & & & & & \\
0 & & & & & & & & \\
0 & & & & & & & &
\end{array}$$

Mixing for $d > 1$

- The **southeast shift** S is the $\mathbb{Z}^2$ SFT with $A = \{0, 1\}$ and $\mathcal{F} = \{ \begin{smallmatrix} 00 \\ 01 \end{smallmatrix} \}$.
- S is block gluing because any two legal rectangular patterns with distance at least 2 can coexist... but the same is not true of non-rectangular patterns, so S is not strongly irreducible.

0	0	0	0	0	0	0	0	0
0	0							
0								
0				1				
0								
0								
0								
0								

Mixing for $d > 1$

- The **southeast shift** S is the $\mathbb{Z}^2$ SFT with $A = \{0, 1\}$ and $\mathcal{F} = \left\{ \begin{smallmatrix} 00 \\ 01 \end{smallmatrix} \right\}$.
- S is block gluing because any two legal rectangular patterns with distance at least 2 can coexist... but the same is not true of non-rectangular patterns, so S is not strongly irreducible.

0	0	0	0	0	0	0	0	0
0	0	0						
0	0							
0				1				
0								
0								
0								
0								

Mixing for $d > 1$

- The **southeast shift** S is the $\mathbb{Z}^2$ SFT with $A = \{0, 1\}$ and $\mathcal{F} = \left\{ \begin{smallmatrix} 00 \\ 01 \end{smallmatrix} \right\}$.
- S is block gluing because any two legal rectangular patterns with distance at least 2 can coexist... but the same is not true of non-rectangular patterns, so S is not strongly irreducible.

$$\begin{array}{lllllllll}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 1 & & & & \\
0 & 0 & 0 & & & & & & \\
0 & 0 & & & & & & & \\
0 & & & & & & & & \\
0 & & & & & & & &
\end{array}$$

Mixing for $d > 1$

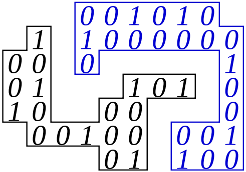
- The $\mathbb{Z}^2$ **hard square shift** $\mathcal{H}$ is the $\mathbb{Z}^2$ SFT with $A = \{0, 1\}$ and $\mathcal{F} = \{11, \substack{1\\1}\}$.

Mixing for $d > 1$

- The $\mathbb{Z}^2$ **hard square shift** $\mathcal{H}$ is the $\mathbb{Z}^2$ SFT with $A = \{0, 1\}$ and $\mathcal{F} = \{11, \begin{smallmatrix}1\\1\end{smallmatrix}\}$.
- $\mathcal{H}$ is strongly irreducible because any legal patterns with distance at least 1 can coexist.

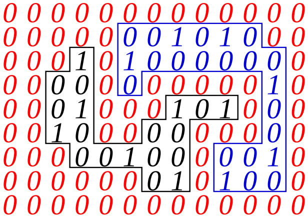
Mixing for $d > 1$

- The $\mathbb{Z}^2$ **hard square shift** $\mathcal{H}$ is the $\mathbb{Z}^2$ SFT with $A = \{0, 1\}$ and $\mathcal{F} = \{11, \begin{smallmatrix}1\\1\end{smallmatrix}\}$.
- $\mathcal{H}$ is strongly irreducible because any legal patterns with distance at least 1 can coexist.



Mixing for $d > 1$

- The $\mathbb{Z}^2$ **hard square shift** $\mathcal{H}$ is the $\mathbb{Z}^2$ SFT with $A = \{0, 1\}$ and $\mathcal{F} = \{11, \frac{1}{1}\}$.
- $\mathcal{H}$ is strongly irreducible because any legal patterns with distance at least 1 can coexist.



Consequences of topological mixing

- Standard topological mixing is a very weak condition

Consequences of topological mixing

- Standard topological mixing is a very weak condition
- (Hochman) There exist topologically mixing $\mathbb{Z}^2$ SFTs with zero entropy

Consequences of topological mixing

- Standard topological mixing is a very weak condition
- (Hochman) There exist topologically mixing $\mathbb{Z}^2$ SFTs with zero entropy
- **Theorem:** (Boyle-P.-Schraudner) There exist topologically mixing $\mathbb{Z}^2$ SFTs with arbitrarily large entropy which do not factor onto any block gluing subshift, and which can contain only a bounded number of disjoint subshifts

Consequences of topological mixing

- Standard topological mixing is a very weak condition
- (Hochman) There exist topologically mixing $\mathbb{Z}^2$ SFTs with zero entropy
- **Theorem:** (Boyle-P.-Schraudner) There exist topologically mixing $\mathbb{Z}^2$ SFTs with arbitrarily large entropy which do not factor onto any block gluing subshift, and which can contain only a bounded number of disjoint subshifts
  - No hope of general SFTs "decomposing" into uniformly mixing pieces

Consequences of topological mixing

- Standard topological mixing is a very weak condition
- (Hochman) There exist topologically mixing $\mathbb{Z}^2$ SFTs with zero entropy
- **Theorem:** (Boyle-P.-Schraudner) There exist topologically mixing $\mathbb{Z}^2$ SFTs with arbitrarily large entropy which do not factor onto any block gluing subshift, and which can contain only a bounded number of disjoint subshifts
  - No hope of general SFTs "decomposing" into uniformly mixing pieces
- Some uniform mixing is necessary for interesting results in $\mathbb{Z}^d$

Consequences of block gluing

- Block gluing implies positive entropy and dense periodic points (PROVE)

Consequences of block gluing

- Block gluing implies positive entropy and dense periodic points (PROVE)
- **Theorem:** (Desai/Boyle-P.-Schraudner) If a block gluing $\mathbb{Z}^d$ SFT has entropy greater than $\log n$, then it factors onto $\{1, \ldots, n\}^{\mathbb{Z}^2}$

Consequences of block gluing

- Block gluing implies positive entropy and dense periodic points (PROVE)
- **Theorem:** (Desai/Boyle-P.-Schraudner) If a block gluing $\mathbb{Z}^d$ SFT has entropy greater than $\log n$, then it factors onto $\{1, \ldots, n\}^{\mathbb{Z}^2}$
- For some purposes, even block gluing does not suffice

Consequences of block gluing

- Block gluing implies positive entropy and dense periodic points (PROVE)
- **Theorem:** (Desai/Boyle-P.-Schraudner) If a block gluing $\mathbb{Z}^d$ SFT has entropy greater than $\log n$, then it factors onto $\{1, \ldots, n\}^{\mathbb{Z}^2}$
- For some purposes, even block gluing does not suffice
- **Theorem:** (P.-Schraudner) For any n, there exists a block gluing $\mathbb{Z}^d$ SFT with entropy $\log n$ which does not factor onto $\{1, \ldots, n\}^{\mathbb{Z}^2}$

Consequences of block gluing

- Block gluing implies positive entropy and dense periodic points (PROVE)
- **Theorem:** (Desai/Boyle-P.-Schraudner) If a block gluing $\mathbb{Z}^d$ SFT has entropy greater than $\log n$, then it factors onto $\{1, \ldots, n\}^{\mathbb{Z}^2}$
- For some purposes, even block gluing does not suffice
- **Theorem:** (P.-Schraudner) For any n, there exists a block gluing $\mathbb{Z}^d$ SFT with entropy $\log n$ which does not factor onto $\{1, \ldots, n\}^{\mathbb{Z}^2}$
- **Theorem:** (Boyle-P.-Schraudner) Block gluing does not imply entropy minimality: there exist block gluing $\mathbb{Z}^2$ SFTs with proper subsystems of equal entropy

Consequences of strong irreducibility

- **Theorem:** (Meester-Steif) Strong irreducibility does imply entropy minimality; any proper subshift of a strongly irreducible $\mathbb{Z}^2$ SFT has strictly smaller entropy

Consequences of strong irreducibility

- **Theorem:** (Meester-Steif) Strong irreducibility does imply entropy minimality; any proper subshift of a strongly irreducible $\mathbb{Z}^2$ SFT has strictly smaller entropy
  - Will prove this later; involves measures!

Consequences of strong irreducibility

- **Theorem:** (Meester-Steif) Strong irreducibility does imply entropy minimality; any proper subshift of a strongly irreducible $\mathbb{Z}^2$ SFT has strictly smaller entropy
  - Will prove this later; involves measures!
- **Theorem:** (Lightwood) If X is a $\mathbb{Z}^2$ SFT with no periodic points, Y is a $\mathbb{Z}^2$ SFT which is strongly irreducible and square filling (didn't define!), and $h(X) < h(Y)$, then X embeds into Y.

Consequences of strong irreducibility

- **Theorem:** (Meester-Steif) Strong irreducibility does imply entropy minimality; any proper subshift of a strongly irreducible $\mathbb{Z}^2$ SFT has strictly smaller entropy
  - Will prove this later; involves measures!
- **Theorem:** (Lightwood) If X is a $\mathbb{Z}^2$ SFT with no periodic points, Y is a $\mathbb{Z}^2$ SFT which is strongly irreducible and square filling (didn't define!), and $h(X) < h(Y)$, then X embeds into Y.
- **Theorem:** (Hochman-Meyerovitch/Friedland) Strongly irreducible SFTs have entropies which are (algorithmically) computable

Consequences of strong irreducibility

- **Theorem:** (Meester-Steif) Strong irreducibility does imply entropy minimality; any proper subshift of a strongly irreducible $\mathbb{Z}^2$ SFT has strictly smaller entropy
  - Will prove this later; involves measures!
- **Theorem:** (Lightwood) If X is a $\mathbb{Z}^2$ SFT with no periodic points, Y is a $\mathbb{Z}^2$ SFT which is strongly irreducible and square filling (didn't define!), and $h(X) < h(Y)$, then X embeds into Y.
- **Theorem:** (Hochman-Meyerovitch/Friedland) Strongly irreducible SFTs have entropies which are (algorithmically) computable
  - More on this next time!