Multidimensional symbolic dynamics

(Minicourse – Lecture 2)

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Preliminaries

$\mathcal{A}$ some finite alphabet (discrete) $\quad d \in \mathbb{N}$ the dimension $(d = 1$ or $d > 1)$

$\mathbb{Z}^d$ **full shift on** $\mathcal{A}$: $\quad \mathcal{A}^{\mathbb{Z}^d}$ (product topology, Cantor) $\quad$ metric: $d(x, y) := 2^{-\inf\{\|\vec{\imath}\|_\infty | \vec{\imath} \in \mathbb{Z}^d:\ x_{\vec{\imath}} \neq y_{\vec{\imath}}\}}$

Configurations are "colorings" of $\mathbb{Z}^d$ using the symbols (letters) in $\mathcal{A}$

Natural $\mathbb{Z}^d$-action: $\sigma : \mathbb{Z}^d \times \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ $\quad$ ($\mathbb{Z}^d$ acting by translations, models d commuting homeomorphisms)

$$\forall\, \vec{\imath}, \vec{\jmath} \in \mathbb{Z}^d, x \in \mathcal{A}^{\mathbb{Z}^d} :\ \sigma(\vec{\imath}, x)_{\vec{\jmath}} = \left(\sigma^{\vec{\imath}}(x)\right)_{\vec{\jmath}} := x_{\vec{\imath}+\vec{\jmath}} \quad \text{(continuous, expansive)}$$

$\mathbb{Z}^d$ **(sub)shifts**: $\quad X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ $\quad$ shift invariant, closed subset $\quad$ (characterization)

Any $\mathbb{Z}^d$ subshift is given by a (countable) family of **forbidden patterns** $\mathcal{F} \subseteq \bigcup_{F \subsetneq \mathbb{Z}^d \text{ finite}} \mathcal{A}^F$ on finite shapes so that

$$X_{\mathcal{F}} := \left\{x \in \mathcal{A}^{\mathbb{Z}^d} \,\middle|\, \forall\, \vec{\imath} \in \mathbb{Z}^d, F \subsetneq \mathbb{Z}^d \text{ finite}:\ x|_{\vec{\imath}+F} \notin \mathcal{F}\right\} = \mathcal{A}^{\mathbb{Z}^d} \setminus \bigcup_{\vec{\imath} \in \mathbb{Z}^d, P \in \mathcal{F}} [P]_{\vec{\imath}}$$

Given X we can recover (find) a family $\mathcal{F}$ such that $X = X_{\mathcal{F}}$: $\quad \mathcal{F} := \bigcup_{F \subsetneq \mathbb{Z}^d \text{ finite}} \{P \in \mathcal{A}^F \mid [P]_{\vec{0}} \cap X = \emptyset\}$

$\mathbb{Z}^d$ **shifts of finite type** (SFTs):

X is a $\mathbb{Z}^d$ SFT $:\Longleftrightarrow\ \exists\, \mathcal{F} \subseteq \bigcup_{F \subsetneq \mathbb{Z}^d \text{ finite}} \mathcal{A}^F$ with $|\mathcal{F}| < \infty$ and $X = X_{\mathcal{F}}$ $\quad$ (local rules, countably many SFTs)

Example: $\quad d := 2,\ \mathcal{A} := \{0, 1\}, \quad \mathcal{F} := \left\{1\,1,\ \begin{matrix}1\\1\end{matrix}\right\}$ $\quad$ ($\mathbb{Z}^2$ golden-mean shift, hard core model)

Wang tiles (1950s) – Nearest-neighbor SFTs

Every $\mathbb{Z}^d$ SFT has a presentation using only restrictions (local rules) $\mathcal{F}$ that constrain pairs of neighboring symbols.
(recoding to a higher-block presentation, representation by a d-tuple of square 0/1 transition matrices)

Equivalently every $\mathbb{Z}^2$ SFT ($\mathbb{Z}^d$ SFT) is given as a **Wang tiling** of $\mathbb{Z}^2$ ($\mathbb{Z}^d$):

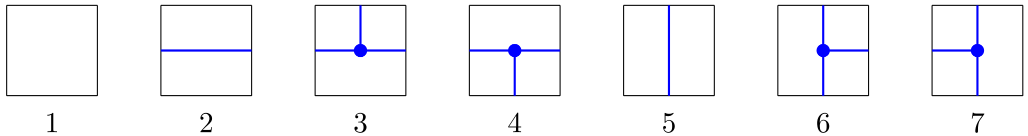
alphabet $\mathcal{A}$: set of 1×1 square (cube) tiles with colored edges (faces)

local rules: adjacent tiles have to have the same color along common edges (faces)

$$\mathcal{F} = \left\{ a\ \ b,\ \begin{matrix} a \\ b \end{matrix} \,\middle|\, a, b \in \mathcal{A} \text{ such that } a,b \text{ have different colors on common edges} \right\}$$

Many examples:

- full shift ($|\mathcal{A}|$ tiles with only one color)
- $\mathbb{Z}^2$ golden-mean shift (uses 5 distinct tiles with two colors: 0 or 1)
- wire shift (7 tiles with two colors: white and blue)



Rules are easy to check – **graphical images** for valid configurations.

For every $N \in \mathbb{N}$ it is also easy to check whether there are $N \times N$ **locally admissible** patterns, but in general it is not obvious to see whether those can be extended to a configuration on all of $\mathbb{Z}^2$ ($\mathbb{Z}^d$). semi-decidable problem

Undecidability results (Berger 1966, Robinson 1971)

In $\mathbb{Z}$ SFTs (given by a matrix or the corresponding graph) simple to decide:

- $\mathbb{Z}$ SFT is not empty $\iff$ matrix does not contain rows/columns of all zeros $\iff$ graph is essential.
- Every **locally admissible** pattern in an essential $\mathbb{Z}$ SFT is **globally admissible** (can be extended to $\mathbb{Z}$).
- Every non-empty $\mathbb{Z}$ SFT contains periodic points.
- Every irreducible $\mathbb{Z}$ SFT (matrix is irreducible, graph is strongly connected) has dense periodic points.

Question of Wang: What about $\mathbb{Z}^d$ SFTs? How to decide whether the shift is empty? (HOPE: periodic points would help!)

In $\mathbb{Z}^d$ shifts a **periodic point** is a point whose orbit under the $\mathbb{Z}^d$ shift-action is finite.

Main problem: Structure of $\mathbb{Z}$ (simple line graph) vs. structure of $\mathbb{Z}^d$ (loops in Cayley graph).

Berger (1966): There are non-empty $\mathbb{Z}^2$ SFTs (even of positive entropy) which do not contain any periodic points.

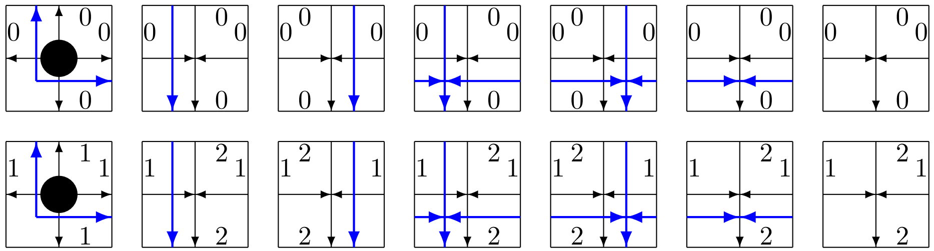
In the class of all $\mathbb{Z}^d$ SFTs ($d > 1$) we are faced with:

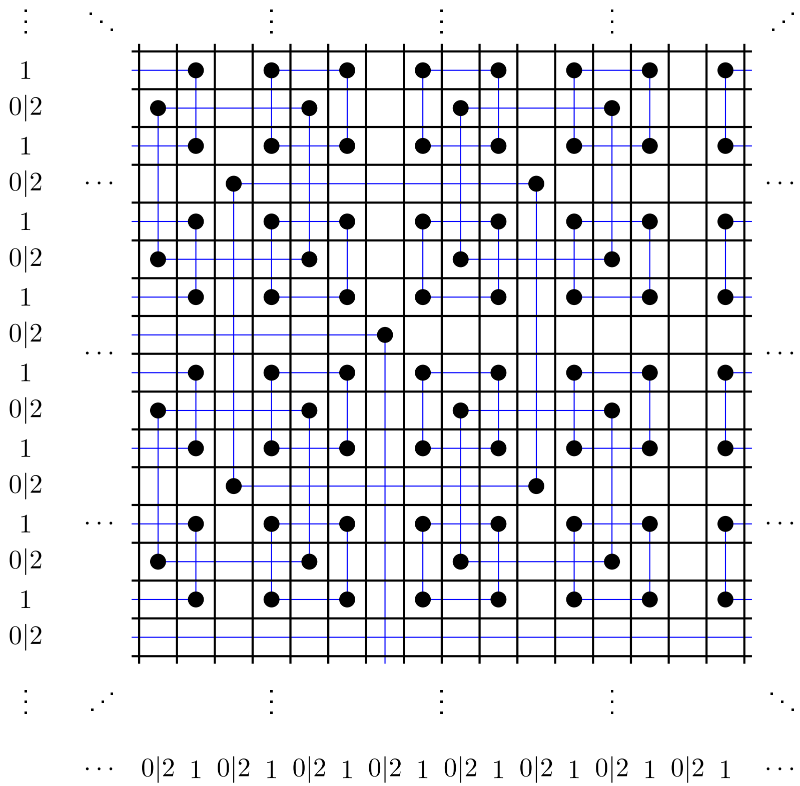
- **emptyness problem**: Given a set of local rules, is the corresponding $\mathbb{Z}^d$ SFT empty? most fundamental
- **extension problem**: Given a locally admissible finite pattern, can it be extended to all of $\mathbb{Z}^d$?
  difference between locally admissible and globally admissible patterns
- **periodic points** (useful in many constructions): Existence or even denseness of periodic points in a $\mathbb{Z}^d$ SFT?

Those questions are **NOT decidable**. No algorithm finishing in finite time! semi-decidable
(Phenomenon not seen in $\mathbb{Z}$ SFTs, caused by Cayley-graph loops.)

Undecidability results – Robinson's tiling (1971)

Robinson gave an example of a $\mathbb{Z}^2$ SFT without periodic points. 56 Wang tiles, points have an hierarchical structure.





Undecidability results – What to do to get around them?

Examples:

- $A_{\text{hor}} := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $A_{\text{vert}} := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ are two essential 0/1-transition matrices. **BUT:** $X_{A_{\text{hor}}, A_{\text{vert}}} = \varnothing$

- $\mathcal{A} := \{0,1\}$, $d := 2$, $\mathcal{F} := \left\{ \begin{smallmatrix} & 0 \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 0 & 0 & 0 \end{smallmatrix}, \begin{smallmatrix} & 0 \\ 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & \\ 0 & 1 \end{smallmatrix}, \ 1\,0\,1 \right\}$ realizes S_{even} along rows, extension problem

In $d = 1$: simple geometry of $\mathbb{Z} \Longrightarrow$ removing finite pattern separates $\mathbb{Z}$-configuration into two pieces
Markov property $\Longrightarrow$ past and future become independent   no flow of information across/around the hole

NOT true in $d > 1$**.**   Main reason for undecidability results: Structure of Cayley graph.

- $\mathbb{Z}^d$ is not just a simple generalization of the $\mathbb{Z}$ setting. Problems are **much harder** in $\mathbb{Z}^d$ SFTs – less results.
- Mixing $\mathbb{Z}$ SFTs are a very homogeneous class of subshifts – most results hold without additional assumptions. $\mathbb{Z}^d$ SFTs are a very **inhomogeneous** class of subshifts; much more **complicated and diverse** than $\mathbb{Z}$ SFTs.
- **No global theory** possible. Instead look for **treatable subclasses** where the undecidability issues vanish.

Not a problem when dealing with examples, but a **serious obstacle** when looking for general results building a theory.

Most results need **additional assumptions**, e.g. mixing conditions.

Mixing vs. Uniform mixing conditions

In $\mathbb{Z}$ most results hold for all mixing $\mathbb{Z}$ SFTs (very homogeneous class + often possible to reduce the general, non-mixing case to the mixing one to get the same or a similar result). apply linear algebra, Perron-Frobenius theory, etc.

A $\mathbb{Z}^d$ subshift X is called: Caused by the geometry of $\mathbb{Z}^d$ mixing splinters into a zoo of possibilities.

- **(topologically) mixing** if for any two non-empty finite subsets $V, W \subsetneq \mathbb{Z}^d$ there exists a constant $D_{V,W} \in \mathbb{N}$ so that for any $\vec{\imath} \in \mathbb{Z}^d$ for which V and $W + \vec{\imath}$ have separation $\delta_\infty(V, W + \vec{\imath}) > D_{V,W}$, any pair of globally admissible patterns on V and $W + \vec{\imath}$ can be put together to form a valid point of X, i.e. for any pair of valid points $x, y \in X$ there exists a valid point $z \in X$ such that $z|_V = x|_V$ and $z|_{W+\vec{\imath}} = y|_{W+\vec{\imath}}$. non-uniform mixing condition
- **block gluing** if there exists a constant $g \in \mathbb{N}$ (a gap size) such that for any two solid blocks $B_1 = [\vec{v}^{(1)}, \vec{w}^{(1)}], B_2 = [\vec{v}^{(2)}, \vec{w}^{(2)}] \subsetneq \mathbb{Z}^d$ with separation $\delta_\infty(B_1, B_2) > g$ and any pair of valid points $x, y \in X$ there exists a valid point $z \in X$ such that $z|_{B_1} = x|_{B_1}$ and $z|_{B_2} = y|_{B_2}$.
- **uniform filling** if there exists a constant $g \in \mathbb{N}$ such that for any solid block $B = [\vec{v}, \vec{w}] \subsetneq \mathbb{Z}^d$ and any pair of valid points $x, y \in X$ there exists a valid point $z \in X$ with $z|_B = x|_B$ and $z|_{\mathbb{Z}^d \setminus [\vec{v} - g\vec{\mathbb{1}}, \vec{w} + g\vec{\mathbb{1}}]} = y|_{\mathbb{Z}^d \setminus [\vec{v} - g\vec{\mathbb{1}}, \vec{w} + g\vec{\mathbb{1}}]}$.
- **strongly irreducible** if there exists a constant $g \in \mathbb{N}$ such that for any pair of non-empty finite subsets $V, W \subsetneq \mathbb{Z}^d$ with separation $\delta_\infty(V, W) > g$ and any pair of valid points $x, y \in X$ there exists a valid point $z \in X$ such that $z|_V = x|_V$ and $z|_W = y|_W$.

Let $\delta_\infty(\vec{v}, \vec{w}) := \max_{1 \leqslant k \leqslant d} |v_k - w_k|$ denote the **maximum-metric** on $\mathbb{Z}^d$ and extend it to non-empty subsets: $\delta_\infty(V, W) := \min_{\vec{v} \in V, \vec{w} \in W} \delta_\infty(\vec{v}, \vec{w})$

Denote by $B = [\vec{v}, \vec{w}] := \mathbb{Z}^d \cap \prod_{k=1}^d [\vec{v}_k, \vec{w}_k]$ a **solid (rectangular/cuboid) block** in $\mathbb{Z}^d$.

For $\mathbb{Z}$ SFTs all these conditions **collapse** to the notion of topological mixing. Probably the reason why this condition is sufficient to built a theory.

In $\mathbb{Z}^d$ SFTs $(d > 1)$ these mixing properties define different (conjugacy-invariant) concepts:

Proposition [Boyle-Pavlov-S]: For $g \in \mathbb{N}$ and any $\mathbb{Z}^d$ shift X we have:

X is strongly irreducible with gap $g \overset{a)}{\Longrightarrow} X$ has the uniform filling property with gap g

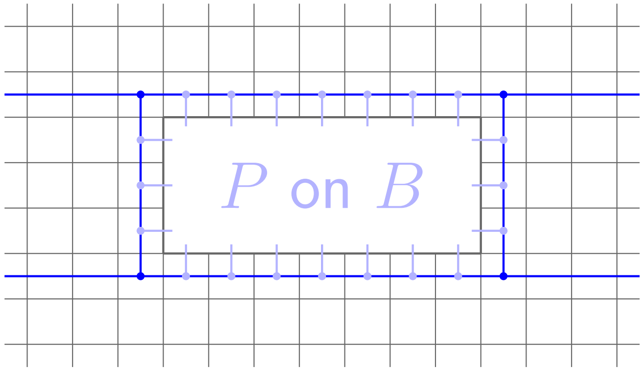
$\overset{b)}{\Longrightarrow} X$ is block gluing with gap $g \overset{c)}{\Longrightarrow} X$ is topologically mixing.

In fact implications $b)$ and $c)$ can not be reversed for $\mathbb{Z}^d$ SFTs, implication $a)$ is not reversible for general $\mathbb{Z}^d$ shifts.

(might be an equivalence for $\mathbb{Z}^d$ SFTs)

Examples:

- full shifts and SFTs with safe symbols like the $\mathbb{Z}^d$ golden-mean shift are strongly irreducible
- the wire shift is block gluing (with gap 2) but does not have the uniform filling property (frozen boundary of all blanks)
- there are topologically mixing $\mathbb{Z}^2$ SFTs with zero entropy (which thus are not block gluing)



Many results desperately need and heavily depend on having **uniform mixing conditions**!

Uniform mixing conditions **exclude pathologies**!

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Topological entropy $\quad X = X_{\mathcal{F}} \subseteq \mathcal{A}^{\mathbb{Z}^d}$ a $\mathbb{Z}^d$ shift on $\mathcal{A}$

$\mathcal{L}^{loc}(X) := \bigcup_{F \subseteq \mathbb{Z}^d \text{ finite}} \left\{ P \in \mathcal{A}^F \mid \forall P' \in \mathcal{F} : \; P' \not\sqsubseteq P \right\}$ **locally admissible** patterns depend on chosen $\mathcal{F}$

$\mathcal{L}(X) := \bigcup_{F \subseteq \mathbb{Z}^d \text{ finite}} \{ x|_F \mid x \in X \}$ **globally admissible** patterns (language)

In general: $\mathcal{L}(X) \subsetneq \mathcal{L}^{loc}(X)$ membership undecidable vs. easy to check

d-**dimensional topological entropy** of a $\mathbb{Z}^d$ subshift X:

$$h_{\text{top}}(X) = \limsup_{n \to \infty} \frac{\log \left| \mathcal{L}_{F(n)}(X) \right|}{|F(n)|}$$

where $F(n) := \left\{ \vec{\jmath} \in \mathbb{Z}^d \;\middle|\; \|\vec{\jmath}\|_\infty < n \right\}$ and $\mathcal{L}_{F(n)}(X) := \left\{ x|_{F(n)} \;\middle|\; x \in X \right\}$ (globally admissible)

Most important numerical invariant of subshifts; measures their complexity in terms of the exponential growth rate of number of globally admissible patterns.

Question: What non-negative real numbers appear as entropies of $\mathbb{Z}^d$ SFTs? (only countably many)

For $\mathbb{Z}$ SFTs (and $\mathbb{Z}$ sofics) simple answer: $h_{\text{top}}(X) = \log \lambda_A$ (Perron eigenvalue, easy to calculate, comes from linear algebra)

Theorem [Lind, 1984]: The topological entropies of $\mathbb{Z}$ SFTs are precisely those non-negative real numbers which are rational multiples of logarithms of Perron numbers. all Perron numbers appear and all entropies are logarithms of algebraic integers

Entropies in the $\mathbb{Z}^d$ setting

No general algorithm (closed formula) to calculate the topological entropy of $\mathbb{Z}^d$ SFTs. (undecidability results of Berger)

Apart from special classes ($\mathbb{Z}^d$ full shifts, $\mathbb{Z}^d$ algebraic shifts, full-$\mathbb{Z}^{d-1}$-extensions of $\mathbb{Z}$ SFTs, zero-entropy $\mathbb{Z}^d$ SFTs etc.)
only very few examples (models from statistical physics, square-ice, dimer, wire-shifts with at least two blanks, certain matrix shifts)
where exact value of topological entropy is known.

Open question: What is the entropy of the $\mathbb{Z}^2$ Fibonacci shift (hard core model)? (only very accurate approximations)

Linear algebra is not available, standard invariants (entropy, zeta-function, same definitions) are a lot less accessible (can not be computed in general) in $\mathbb{Z}^d$ SFTs. Nevertheless we have a complete theoretical characterization of entropy:

Theorem [Hochman-Meyerovitch, 2007]: For $d \geqslant 2$ the class of topological entropies of $\mathbb{Z}^d$ SFTs coincides with the class of **right-recursively-enumerable** non-negative real numbers. the same is true for $\mathbb{Z}^d$ sofics

Definition:

A real number $r \in \mathbb{R}$ is **computable** if there exists a Turing machine that given input $n \in \mathbb{N}$ outputs a rational approximation $r_n \in \mathbb{Q}$ of r such that $\left|r - r_n\right| \leqslant \frac{1}{n}$.

A real number $r \in \mathbb{R}$ is **right-recursively-enumerable** if there exists a Turing machine that given input $n \in \mathbb{N}$ outputs a rational approximation $r_n \in \mathbb{Q}$ of r such that the sequence $r_n \searrow r$. no error bounds

strictly contains the class of computable numbers and contains some non-algebraic numbers (e.g. power series for e and π with known error bounds)

A word on the proof of Hochman-Meyerovitch

Not hard to show that entropies are always right-recursively-enumerable. count number of locally admissible patterns

Intricate construction to realize all of those numbers as entropies using tools from **recursion theory**:

- Construct a multi-layer, zero-entropy $\mathbb{Z}^d$ SFT where the density of certain symbols is very regular over all points (uses Toeplitz-type structure like in the Robinson tiling)
- Use Turing machines running on certain regions in a superimposed layer to check those densities and forbid points where the density is to high pruning
- Introduce independent copies of those symbols to generate positive entropy in a controlled way (only known possibility to increase entropy by an exact amount)

But: The proposed construction is quiet **rigid**:

- there is always a non-trivial zero-entropy factor (colapsing back the independent copies)
- there is no kind of mixing (one superimposed layer is just constant in all but one direction)
- not even topologically transitive

Questions: Can we get a less degenerate construction?

What about subclasses of $\mathbb{Z}^d$ SFTs which have some kind of (uniform) mixing?

Entropies of uniformly mixing $\mathbb{Z}^d$ SFTs

Proposition [Boyle-Pavlov-S]:

(1) For $\mathbb{Z}^2$ shifts: Block gluing $\Longrightarrow$ denseness of periodic points (open for $\mathbb{Z}^d$ with $d > 2$ even in the strongly irreducible setting)

(2) Block gluing, non-trivial $\Longrightarrow$ positive entropy

(3) Block gluing $\Longrightarrow$ no non-trivial zero-entropy factor

(4) Block gluing $\not\Longrightarrow$ entropy minimality (e.g. wire shifts with at least two blanks)

(5) Uniform filling property $\Longrightarrow$ entropy minimality

A $\mathbb{Z}^d$ subshift Y is a **factor** of X $:\Longleftrightarrow \exists\, \phi:\ X \to Y$ surjective continuous map such that $\phi \circ \sigma_X = \sigma_Y \circ \phi$

A $\mathbb{Z}^d$ subshift X is **entropy minimal** if any proper subshift Y of X has strictly less entropy, i.e. $h_{\text{top}}(Y) < h_{\text{top}}(X)$

Question: What about the values of topological entropy for uniformly mixing $\mathbb{Z}^d$ SFTs?

Good **approximation algorithms** using strip entropies (Markley-Paul, Friedland, Pavlov). up to exponential convergence

First theoretical result:

Theorem [Hochman-Meyerovitch, 2007]: For $d \geqslant 1$ the topological entropy of any strongly irreducible $\mathbb{Z}^d$ SFT has to be a computable non-negative real number. Much stronger condition, but still involving recursion theory.

Question [same paper]: Is this condition sufficient? What numbers are actually realizable?

Entropies of uniformly mixing $\mathbb{Z}^d$ SFTs – A stronger necessary condition

Definition: Let $(t_n \in \mathbb{N})_{n\in\mathbb{N}}$ be a non-decreasing sequence of natural numbers. A real number $\alpha \in \mathbb{R}$ is **computable with rate** $(t_n)_{n\in\mathbb{N}}$ if there exists a (deterministic) Turing machine, which for any $n \in \mathbb{N}$ outputs in at most t_n steps a rational approximation $\frac{r_n}{s_n} \in \mathbb{Q}$ of α such that $\left|\alpha - \frac{r_n}{s_n}\right| \leqslant \frac{1}{n}$.

Theorem [Pavlov-S]: For any block gluing $\mathbb{Z}^2$ shift of finite type X there exists some constant $C \in \mathbb{R}$ such that its topological entropy $h_{\text{top}}(X)$ is computable with rate $\left(2^{C\cdot n^2}\right)_{n\in\mathbb{N}}$.

Remark: A slightly weaker restriction holds for $\mathbb{Z}^d$ SFTs having the uniform filling property and a periodic point.

Idea of proof: Relate number of locally admissible patterns to number of globally admissible patterns and use:

$$\frac{\log|\mathcal{L}_{m,m}(X)|}{(m+g)^2} \leqslant h_{\text{top}}(X) \leqslant \frac{\log|\mathcal{L}_{m,m}(X)|}{m^2} \qquad \text{(block gluing)}$$

to get an error bound on the goodness of the entropy approximation:

$$\left|h_{\text{top}}(X) - \frac{\log|\mathcal{L}_{m,m}(X)|}{m^2}\right| \leqslant \log|\mathcal{L}_{m,m}(X)| \cdot \left(\frac{1}{m^2} - \frac{1}{(m+g)^2}\right) \leqslant \frac{2g\log|\mathcal{L}_{m,m}(X)|}{(m+g)\cdot m^2} \leqslant \frac{2g\log|\mathcal{A}|}{m}$$

For block gluing $\mathbb{Z}^2$ SFTs there is a deterministic (brute-force) algorithm to compute the number $|\mathcal{L}_{m,m}(X)|$ of globally admissible patterns in $|\mathcal{A}|^{C'\cdot m^2} \leqslant 2^{C\cdot n^2}$ steps.

Entropies of uniformly mixing $\mathbb{Z}^d$ SFTs – A sufficient condition (1)

Definition: A non-negative real number $\alpha \in \mathbb{R}$ **satisfies the computability condition (C)** if it allows a representation $\alpha = \alpha' \cdot \log M$ for some natural number $M \in \mathbb{N}$ and some $\alpha' \in \mathbb{R}$ which is either rational or which has an infinite continued fraction expansion $\alpha' = [a_0; a_1, a_2, a_3, \ldots]$ such that there is a (sped-up) Turing machine which for every $N \in \mathbb{N}$ calculates a (binary) representation of the first N partial quotients $a_1, a_2, a_3, \ldots, a_N$ in at most $a_N \cdot t_{N-1}$ steps where the sequence $(t_n \in \mathbb{N})_{n\in\mathbb{N}_0}$ is recursively defined by $t_0 := 1$, $t_1 := a_1$ and $t_n := a_n \cdot t_{n-1} + t_{n-2}$ for all $2 \leqslant n \in \mathbb{N}$.

Examples: All non-negative real numbers $\alpha = \alpha' \cdot \log M \ (M \in \mathbb{N})$ with

- $\alpha' \in \mathbb{Q}$ an arbitrary rational number or
- $\alpha' \in \mathbb{R}\backslash\mathbb{Q}$ an irrational number with an
  1. **eventually constant** continued fraction expansion (e.g. numbers like $\frac{\sqrt{5}-1}{2} = [0; 1, 1, 1, 1, \ldots]$ and $\frac{\sqrt{2}}{2} = [0; 1, 2, 2, 2, \ldots]$ etc.)
  2. **eventually periodic** continued fraction expansion (i.e. all irrational quadratic algebraic numbers)
  3. **eventually affine** continued fraction expansion where there exist two constants $C \in \mathbb{N}$ and $D \in \mathbb{Z}$ and a starting point $N \in \mathbb{N}$ such that for all $N \leqslant n \in \mathbb{N}$ the partial quotients are of the form $a_n = C \cdot n + D$
  4. **eventually affine-periodic** continued fraction expansion where there exist $2p$ constants $C_0, C_1, \ldots, C_{p-1} \in \mathbb{N}$ and $D_0, D_1, \ldots, D_{p-1} \in \mathbb{Z}$ and a starting point $N \in \mathbb{N}$ such that for all $N \leqslant n \in \mathbb{N}$ the partial quotients are of the form $a_n = C_{n \bmod p} \cdot n + D_{n \bmod p}$

satisfy the computability condition **(C)**.

Classes (1) to (4) are nested but strictly increasing.

Entropies of uniformly mixing $\mathbb{Z}^d$ SFTs – A sufficient condition (2)

Classes (3) and (4) in particular include well known (transcendental) numbers like:

$$\begin{aligned}
\tanh\left(\tfrac{1}{k}\right) &= [0; k, 3k, 5k, 7k, \ldots] \quad \text{with } a_n := 2k \cdot n - k \text{ and } k \in \mathbb{N} \text{ arbitrary.} \\
\tan(1) &= [1; 1, 1, 3, 1, 5, 1, 7, 1, \ldots] \\
\tan\left(\tfrac{1}{k}\right) &= [0; k-1, 1, 3k-2, 1, 5k-2, 1, 7k-2, 1, \ldots] \quad (k > 1) \\
\exp(1) &= [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \ldots] \quad \text{or} \\
\exp\left(\tfrac{1}{k}\right) &= [1; k-1, 1, 1, 3k-1, 1, 1, 5k-1, 1, 1, 7k-1, 1, 1, \ldots] \quad (k > 1)
\end{aligned}$$

Theorem [Pavlov-S]: Suppose $d \geqslant 3$. For any non-negative real number $\alpha \in \mathbb{R}_0^+$ which satisfies the computability condition **(C)**, there exists a block gluing $\mathbb{Z}^d$ SFT X with topological entropy $h_{\text{top}}(X) = \alpha$.

Remark: We expect the result to extend to $d = 2$, but the construction will be more complicated.

Idea of proof:

Our construction refines techniques of Hochman-Meyerovitch using **Robinson's tiling**, **Turing machines** and **Sturmian subshifts** to control not only frequencies of symbols but also number of locally admissible patterns.

Again this would only give rigid non-mixing examples but by our more careful construction those can be **upgraded** to be block gluing without affecting the entropy thus giving **uniform mixing** and **avoiding zero-entropy factors**.

Summary of this lecture

- Definitions and questions of $\mathbb{Z}$ symbolic dynamics can be generalized naturally to the $\mathbb{Z}^d$ framework.
- However the answers and results are very different from the classical theory – properties generalize if at all only to certain subclasses*.
- The useful structures of $\mathbb{Z}$ SFTs – matrices, graphs, algebraic invariants – are not accessible in $\mathbb{Z}^d$.
- The world of multidimensional $\mathbb{Z}^d$ SFTs $(d > 1)$ is more varied, vastly richer and much more complicated than the class of $\mathbb{Z}$ SFTs.
- There is no global theory for the very inhomogeneous class of $\mathbb{Z}^d$ SFTs, but many results for certain subclasses.
- Undecidability and recursion theory plays a large role – construction techniques rely on Turing machines.
- Uniform mixing conditions help in avoiding pathologies and undecidability issues.
- We are still exploring the "landscape", finding interesting examples, discovering new properties and unexpected phenomena.

*The strongest, most developed $\mathbb{Z}^d$ theory exists for algebraic group shifts built over an alphabet which is a finite abelian group using Pontryagin duality principles and the structure of Noetherian modules. Work by Schmidt, Einsiedler, Ward, ...