Multidimensional symbolic dynamics

(Minicourse – Lecture 2)

Michael H. Schraudner

Centro de Modelamiento Matemático
Universidad de Chile

mschraudner@dim.uchile.cl

www.cmm.uchile.cl/∼mschraudner

1st French-Chilean conference in Dynamics and Combinatorics

January 25th, 2012 – Cap Hornu, Baie de Somme

Entropies of $\mathbb{Z}^d$ SFTs (continued)

Recall:

Theorem [Hochman-Meyerovitch, 2007]: For $d \geqslant 2$ the class of topological entropies of $\mathbb{Z}^d$ SFTs coincides with the class of **right-recursively-enumerable** non-negative real numbers.

the same is true for $\mathbb{Z}^d$ sofics

But: The proposed construction is quiet **rigid**:

- there is always a non-trivial zero-entropy factor
- there is no kind of mixing (one layer is constant in all but one direction)
- not even topologically transitive

Questions: Can we get a less degenerate construction?

What about subclasses of $\mathbb{Z}^d$ SFTs which have some kind of (uniform) mixing?

Entropies of uniformly mixing $\mathbb{Z}^d$ SFTs

Proposition [Boyle-Pavlov-S]:

(1) For $\mathbb{Z}^2$ shifts: Block gluing $\Longrightarrow$ denseness of periodic points
(open for $\mathbb{Z}^d$ with $d > 2$ even in the strongly irreducible setting)

(2) Block gluing, non-trivial $\Longrightarrow$ positive entropy

(3) Block gluing $\Longrightarrow$ no non-trivial zero-entropy factor

(4) Block gluing $\not\Longrightarrow$ entropy minimality (e.g. wire shifts with at least two blanks)

(5) Uniform filling property $\Longrightarrow$ entropy minimality

A $\mathbb{Z}^d$ subshift Y is a **factor** of X $:\Longleftrightarrow \exists\, \phi: \; X \to Y$ surjective continuous map such that $\phi \circ \sigma_X = \sigma_Y \circ \phi$

A $\mathbb{Z}^d$ subshift X is **entropy minimal** if any proper subshift Y of X has strictly less entropy, i.e. $h_{\mathrm{top}}(Y) < h_{\mathrm{top}}(X)$

Question: What about the values of topological entropy for uniformly mixing $\mathbb{Z}^d$ SFTs?

Good **approximation algorithms** using strip entropies (Markley-Paul, Friedland, Pavlov).

up to exponential convergence

Entropies of uniformly mixing $\mathbb{Z}^d$ SFTs – Necessary conditions

First theoretical result:

Theorem [Hochman-Meyerovitch, 2007]: For $d \geqslant 1$ the topological entropy of any strongly irreducible $\mathbb{Z}^d$ SFT has to be a computable non-negative real number.

Much stronger condition, less pathological numbers, but still involving recursion theory.

Definition: Let $(t_n \in \mathbb{N})_{n \in \mathbb{N}}$ be a non-decreasing sequence of natural numbers. A real number $r \in \mathbb{R}$ is **computable with rate** $(t_n)_{n \in \mathbb{N}}$ if there exists a (deterministic) Turing machine, which for any $n \in \mathbb{N}$ outputs in at most t_n steps a rational approximation $q_n \in \mathbb{Q}$ of r such that $\left|r - q_n\right| \leqslant \frac{1}{n}$.

Theorem [Pavlov-S]: For any block gluing $\mathbb{Z}^2$ shift of finite type X there exists some constant $C \in \mathbb{R}$ such that its topological entropy $h_{\text{top}}(X)$ is computable with rate $\left(2^{C \cdot n^2}\right)_{n \in \mathbb{N}}$.

Remark: A slightly weaker restriction holds for $\mathbb{Z}^d$ SFTs having the UFP and a periodic point.

Question: Is this condition sufficient? What numbers are actually realizable?

Entropies of uniformly mixing $\mathbb{Z}^d$ SFTs – A sufficient condition

Definition: A non-negative real number $\alpha \in \mathbb{R}$ **satisfies the computability condition (C)** if it allows a representation $\alpha = \alpha' \cdot \log M$ for some natural number $M \in \mathbb{N}$ and some $\alpha' \in \mathbb{R}$ which is either rational or which has an infinite continued fraction expansion $\alpha' = [a_0; a_1, a_2, a_3, \ldots]$ such that there is a (sped-up) Turing machine which for every $N \in \mathbb{N}$ calculates a (binary) representation of the first N partial quotients $a_1, a_2, a_3, \ldots, a_N$ in at most $a_N \cdot t_{N-1}$ steps where the sequence $(t_n \in \mathbb{N})_{n \in \mathbb{N}_0}$ is recursively defined by $t_0 := 1$, $t_1 := a_1$ and $t_n := a_n \cdot t_{n-1} + t_{n-2}$ for all $2 \leqslant n \in \mathbb{N}$.

Theorem [Pavlov-S]: Suppose $d \geqslant 3$. For any non-negative real number $\alpha \in \mathbb{R}_0^+$ which satisfies the computability condition **(C)**, there exists a block gluing $\mathbb{Z}^d$ SFT X with topological entropy $h_{\text{top}}(X) = \alpha$.

Remarks:

- We expect the result to extend to $d = 2$, but the construction will be more complicated.
- Construction (explain) produces **non-entropy minimal** block gluing $\mathbb{Z}^d$ SFT.
- Construction **not possible** for stronger mixing conditions (UFP or strongly irreducible).

Examples: All non-negative real numbers $\alpha = \alpha' \cdot \log M \ (M \in \mathbb{N})$ with

- $\alpha' \in \mathbb{Q}$ an arbitrary rational number or
- $\alpha' \in \mathbb{R}\backslash\mathbb{Q}$ an irrational number with an

(1) **eventually constant** continued fraction expansion (e.g. $\frac{\sqrt{5}-1}{2} = [0; 1, 1, 1, 1, \ldots]$ and $\frac{\sqrt{2}}{2} = [0; 1, 2, 2, 2, \ldots]$ etc.)

(2) **eventually periodic** continued fraction expansion (i.e. all irrational quadratic algebraic numbers)

(3) **eventually affine-periodic** continued fraction expansion where there exist $2p$ constants $C_0, C_1, \ldots, C_{p-1} \in \mathbb{N}$ and $D_0, D_1, \ldots, D_{p-1} \in \mathbb{Z}$ and a starting point $N \in \mathbb{N}$ such that for all $N \leqslant n \in \mathbb{N}$ the partial quotients are of the form $a_n = C_{n \bmod p} \cdot n + D_{n \bmod p}$

satisfy the computability condition **(C)**.

Class (3) in particular contains well known (non-algebraic) numbers like:

$$\begin{aligned}
\tanh\left(\tfrac{1}{k}\right) &= [0; k, 3k, 5k, 7k, \ldots] \quad \text{with } a_n := 2k \cdot n - k \text{ and } k \in \mathbb{N} \text{ arbitrary.} \\
\tan(1) &= [1; 1, 1, 3, 1, 5, 1, 7, 1, \ldots] \\
\tan\left(\tfrac{1}{k}\right) &= [0; k-1, 1, 3k-2, 1, 5k-2, 1, 7k-2, 1, \ldots] \quad (k > 1) \\
\exp(1) &= [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \ldots] \quad \text{or} \\
\exp\left(\tfrac{1}{k}\right) &= [1; k-1, 1, 1, 3k-1, 1, 1, 5k-1, 1, 1, 7k-1, 1, 1, \ldots] \quad (k > 1)
\end{aligned}$$

SFTs, sofic shifts and their languages

A $\mathbb{Z}^d$ subshift S is called **sofic** iff it is the factor of some $\mathbb{Z}^d$ SFT X, i.e. $\exists\, \phi :\ X \twoheadrightarrow S$

continuous shift-commuting surjection

Every $\mathbb{Z}^d$ SFT is itself a $\mathbb{Z}^d$ sofic, but there are **proper** $\mathbb{Z}^d$ sofics which are not $\mathbb{Z}^d$ SFTs.

All $\mathbb{Z}$ sofics can be represented by a finite directed **labeled graph**.

Recall:

$\mathcal{L}(X) := \bigcup_{F \subseteq \mathbb{Z}^d \text{ finite}} \{x|_F \mid x \in X\}$ **globally admissible** patterns (language)

For $\mathbb{Z}$ SFTs: $\mathcal{L}(X) = \mathcal{A}^* \backslash (\mathcal{A}^* \mathcal{F} \mathcal{A}^*)$ special regular language

For $\mathbb{Z}$ sofics: $\mathcal{L}(S)$ is a (factorial, extensible) regular language (in fact a characterization of $\mathbb{Z}$ sofics)

As regular languages are at the base of the Chomsky hierarchy, this means that $\mathbb{Z}$ sofics are combinatorially "simple" systems.

Question: What about the languages of $\mathbb{Z}^d$ SFTs or $\mathbb{Z}^d$ sofics?

Effective subshifts

Any $\mathbb{Z}^d$ subshift is given by a (countable) family of **forbidden patterns** $\mathcal{F} \subseteq \bigcup_{F \subsetneq \mathbb{Z}^d \text{ finite}} \mathcal{A}^F$ on finite shapes so that

$$X_{\mathcal{F}} := \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} \;\middle|\; \forall\, \vec{\imath} \in \mathbb{Z}^d, F \subsetneq \mathbb{Z}^d \text{ finite}: \; x|_{\vec{\imath}+F} \notin \mathcal{F} \right\} = \mathcal{A}^{\mathbb{Z}^d} \setminus \bigcup_{\vec{\imath} \in \mathbb{Z}^d, P \in \mathcal{F}} [P]_{\vec{\imath}}$$

A $\mathbb{Z}^d$ subshift X is called **effective** if there exists a **recursively enumerable** set $\mathcal{F}$ such that $X = X_{\mathcal{F}}$.

Observations:

- Effectiveness is preserved under conjugacy
- $\mathbb{Z}^d$ SFTs are effective   finite sets are recursively enumerable
- $\mathbb{Z}^d$ sofic shifts are effective   consequence of Hochman (2009): symbolic factors of effective systems are effective

In fact most $\mathbb{Z}^d$ subshift examples are effective by default.

But: only countably many effective $\mathbb{Z}^d$ subshifts – hence the **big majority is not effective**

Examples: Sturmian subshifts   effectiveness depends on the rotation angle $\alpha \in [0, 1)$ being effective

Michael Schraudner mschraudner@dim.uchile.cl

Hochman's subdynamics

As $\mathbb{Z}^d$ shifts are complicated objects, **why not try to understand their $\mathbb{Z}$-"subsystems"?**

Let $(X, (T_g)_{g\in G})$ be a dynamical system with some G-action and let $H \leqslant G$ be a subgroup.

think of $G = \mathbb{Z}^d$

The system $(X, (T_g)_{g\in H})$ is called an H-**subaction**. (here we only restrict the action but keep the phase space)

Examples:

- constant-$\mathbb{Z}^{d-1}$-extension of a $\mathbb{Z}$ subshift Y: The $\mathbb{Z}$-subaction (along the first base vector) is topologically conjugate to Y.
- full-$\mathbb{Z}^{d-1}$-extension of a non-trivial $\mathbb{Z}$ subshift Y: $\mathbb{Z}$-subaction is not a subshift (expansive).

uncountable alphabet

Theorem [Hochman (2009), Aubrun-Sablik (2010)]:

Any effective $\mathbb{Z}^d$-subshift can be seen as the subdynamics of some $\mathbb{Z}^{d+1}$ sofic shift.
The class of possible subactions coincides with the class of effective subshifts.

Language can be very complex (high up in the Chomsky hierarchy).

Theorem [Hochman (2009), Aubrun-Sablik (2010)]:

Any effective $\mathbb{Z}^d$-subshift can be seen as the subdynamics of some $\mathbb{Z}^{d+1}$ sofic shift.

The class of possible subactions coincides with the class of effective subshifts.

Remarks:

- $\mathbb{Z}^d$ SFTs and sofics are effective. There symbolic subactions have to be effective again.
- Effectiveness is the only constraint.
  - This gives rise to a rich and diverse landscape of possible subactions.
  - However this also means: Every bad thing that could occur, will in fact happen.
- Not every effective $\mathbb{Z}^d$ subshift is sofic. The increase in dimension is needed and optimal.
  extra dimension is used for implementing a Turing machine that produces forbidden patterns and checks those do not appear
- The result does not hold for $\mathbb{Z}^{d+1}$ SFTs. Construction needs a little more space. extra layers
- The cover (difference between $\mathbb{Z}^d$ sofic and $\mathbb{Z}^d$ SFT) is measure-theoretically small. adds rational spectrum
- Again the construction is very rigid (constant along a subaction) and does not allow for mixing.

Open problem: When exactly is the $\mathbb{Z}^d$ sofic needed? Characterize the subactions of $\mathbb{Z}^d$ SFTs.

Projective subdynamics

Define k-dimensional **sublattices** of $\mathbb{Z}^d$. (for this lecture think of $k = 1$)

$\mathbb{Z}^d$ shifts contain subshifts of smaller dimension obtained by projecting onto k-dimensional sublattices of $\mathbb{Z}^d$.

Let X be a $\mathbb{Z}^d$ subshift and $L \lneq \mathbb{Z}^d$ a k-dimensional sublattice of $\mathbb{Z}^d$ $(k < d)$

$$\mathsf{P}_L(X) := \left\{ x|_L \;\middle|\; x \in X \right\} \qquad \text{the } L\text{-}\textbf{projective subdynamics} \text{ of } X$$

As $L \cong \mathbb{Z}^k$ we have a natural $\mathbb{Z}^k$-shift action on $\mathsf{P}_L(X)$: $\sigma|_{L \times \mathsf{P}_L(X)} : L \times \mathsf{P}_L(X) \to \mathsf{P}_L(X)$

This time we restrict the action and the phase space – difference to Hochman's subdynamics – advantage: always a subshift

Try to understand the complicated $\mathbb{Z}^d$ dynamics by looking at the lower dimensional projective subdynamics.

Question: Given a class of $\mathbb{Z}^d$ shifts (e.g. $\mathbb{Z}^d$ SFTs), what $\mathbb{Z}^k$ shifts appear as L-projective subdynamics?

Stable vs. unstable projective subdynamics

Let $X = X_{\mathcal{F}}$ be some $\mathbb{Z}^d$ subshift with forbidden patterns $\mathcal{F}$ and $L \lneq \mathbb{Z}^d$ a k-dimensional sublattice.

X defines a decreasing sequence of $\mathbb{Z}^k$ subshifts $\big(X_{L,n} \subseteq \mathcal{A}^L\big)_{n\in\mathbb{N}_0}$ with $X_{L,n+1} \subseteq X_{L,n}$

$$X_{L,n} := \Big\{x|_L \;\Big|\; x \in \mathcal{A}^{L\times[-n,n]^{d-k}} \wedge \forall\, F \subsetneq L \times [-n,n]^{d-k} \text{ finite}:\ x|_F \notin \mathcal{F}\Big\}$$

We have: $\mathsf{P}_L(X) = \bigcap_{n\in\mathbb{N}_0} X_{L,n}$.

Definition: The L-projective subdynamics $\mathsf{P}_L(X)$ of X is called

stable $:\Longleftrightarrow \exists\, N \in \mathbb{N}_0\ \forall\, n \geqslant N$: $X_{L,n} = X_{L,N} = \mathsf{P}_L(X)$ (sequence stabilizes)

unstable $:\Longleftrightarrow \forall\, n \in \mathbb{N}_0\ \exists\, n' > n$: $X_{L,n'} \subsetneq X_{L,n}$ (sequence is infinitely decreasing)

Generalization of notion of (un-) stable limit set of CAs [Maass, 1995].

Remark: Whenever a $\mathbb{Z}^k$ subshift Y appears as

(un-) stable projective subdynamics in some $\mathbb{Z}^d$ SFT (sofic, ...) then it also appears as (un-) stable projective subdynamics in some $\mathbb{Z}^{d'}$ SFT (sofic, ...) for any $d' > d$.

Use trivial full-$\mathbb{Z}^{d'-d}$-extensions.

Question: What is the smallest dimension d in which certain $\mathbb{Z}^k$ subshifts can be realized as k-dimensional projective subdynamics?

In the following we concentrate on $\mathbb{Z}^d$ SFTs asking what $\mathbb{Z}^k$ subshifts **do** or **do not** appear as their projective subdynamics.

How can local rules in d-dimensions be used to get complicated, non-local $\mathbb{Z}^k$ behaviour?

In particular the case $k = 1$ is feasible and we obtain a **classification** of realizable $\mathbb{Z}$ sofics.

As $L \cong \mathbb{Z}^k$ we may consider $L = \langle \vec{e}_1, \vec{e}_2, \dots, \vec{e}_k \rangle_{\mathbb{Z}} = \mathbb{Z}^k$ which is sufficient due to the following:

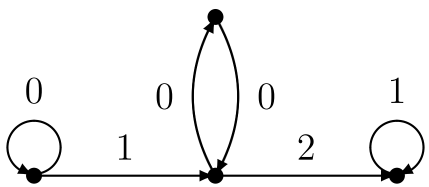
Remark: Let $L \lneq \mathbb{Z}^d$ be any k-dimensional sublattice and $Y \subseteq \mathcal{A}^L$ be a $\mathbb{Z}^k$ subshift on $\mathcal{A}$.
There exists a $\mathbb{Z}^d$ SFT $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ with $\mathsf{P}_L(X) = Y$ if and only if there exists a $\mathbb{Z}^d$ SFT $\widetilde{X} \subseteq \mathcal{A}^{\mathbb{Z}^d}$ with $\mathsf{P}_{\mathbb{Z}^k}(\widetilde{X}) = Y$ and the relation between X and $\widetilde{X}$ is highly constructive.

Two conditions governing realizability

A $\mathbb{Z}^d$ subshift X has **universal period** if there exists a global bound $M \in \mathbb{N}$ so that for every point $x \in X$ there is a finite set $F_x \subsetneq \mathbb{Z}^d$ of coordinates with $|F_x| < M$ and a point $y \in \mathrm{Per}(X)$ (depending on x) such that $x|_{\mathbb{Z}^d \setminus F_x} = y|_{\mathbb{Z}^d \setminus F_x}$. points look like periodic points except on a bounded number of symbols

Universal period vs. No universal period

A zero entropy $\mathbb{Z}$ sofic shift S has a **good set of periods**, if it allows a (right-resolving) graph presentation G such that for every cycle length $l \in \mathbb{N}$ present in G there exists a finite collection of periodic points in S such that the least common multiple of all their least periods is a multiple of l.



No good set of periods!

Theorem (Sofic projective subdynamics) [Pavlov-S]: For any $k < d \in \mathbb{N}$, we have:

- Results on stable projective subdynamics:
  - Every $\mathbb{Z}^k$ SFT is the stable k-dimensional projective subdynamics of some $\mathbb{Z}^d$ SFT. trivially
  - The stable k-dimensional projective subdynamics of any $\mathbb{Z}^d$ SFT has to be a $\mathbb{Z}^k$ sofic shift.
  - Any proper $\mathbb{Z}$ sofic with **positive entropy** can be realized as the stable projective subdynamics of some $\mathbb{Z}^2$ SFT (hence also of some $\mathbb{Z}^d$ SFT for any $d > 2$).
  - A **zero-entropy** proper $\mathbb{Z}$ sofic is realizable as the stable projective subdynamics of some $\mathbb{Z}^2$ SFT if and only if it has **a good set of periods** but **no universal period**.
- Results on unstable projective subdynamics:
  - No $\mathbb{Z}^k$ SFT is realizable as the unstable k-dimensional projective subdynamics of any $\mathbb{Z}^d$ SFT.
  - Any proper $\mathbb{Z}$ sofic with **positive entropy** can be realized as the unstable projective subdynamics of some $\mathbb{Z}^2$ SFT (hence also of some $\mathbb{Z}^d$ SFT for any $d > 2$).
  - A **zero-entropy** proper $\mathbb{Z}$ sofic is realizable as the unstable projective subdynamics of some $\mathbb{Z}^2$ SFT if and only if it **does not have universal period**.

Positive entropy $\mathbb{Z}$ subshifts are "easy" to realize as projective subdynamics, difficulties arise from zero-entropy $\mathbb{Z}$ subshifts.

Uniformly mixing $\mathbb{Z}^d$ SFTs force sofic projective subdynamics

The previous constructions realizing $\mathbb{Z}$ subshifts as projective subdynamics in $\mathbb{Z}^d$ SFTs did not care about mixing conditions and in general produced rather rigid $\mathbb{Z}^d$ SFTs.

Question: What $\mathbb{Z}$ subshifts can be realized in $\mathbb{Z}^d$ SFTs with strong mixing properties?

Theorem (Projective subdynamics in uniformly mixing $\mathbb{Z}^d$ SFTs) [S]:

- There are $\mathbb{Z}^2$ (and $\mathbb{Z}^d$) SFTs which are **topologically mixing** even with linearly growing mixing distance but which still show highly non-sofic $\mathbb{Z}$-projective subdynamics (along their cardinal directions).

- The one-dimensional projective subdynamics along cardinal directions of any **block gluing** $\mathbb{Z}^2$ SFT has to be a mixing $\mathbb{Z}$ sofic. However this sofic projective subdynamics does not have to be stable. Moreover block gluing $\mathbb{Z}^2$ SFTs can still have non-sofic one-dimensional projective subdynamics along non-cardinal directions.

- The one-dimensional projective subdynamics along cardinal directions in any $\mathbb{Z}^d$ SFT with the **uniform filling property** has to be a mixing $\mathbb{Z}$ sofic. The same holds along arbitrary directions under the assumption of **strong irreducibility** of the $\mathbb{Z}^d$ SFT.

- Any mixing $\mathbb{Z}$ sofic can be realized as the $\mathbb{Z}$-projective subdynamics in a **strongly irreducible $\mathbb{Z}^2$ SFT** (hence also in a strongly irreducible $\mathbb{Z}^d$ SFT for any $d > 2$).

$\Longrightarrow$ Hence **uniform mixing** is a **severe condition** limiting (not only) possible projective subdynamics.

Non-realizable non-sofic one-dimensional projective subdynamics

Theorem [Pavlov-S]:

Let Z be a $\mathbb{Z}$ subshift **without periodic points** such that there exist arbitrarily long words $w_k \in \mathcal{L}(Z)$ and positive integers $n_k \in \mathbb{N}$ ($k \in \mathbb{N}$) with the property that for every k, every $z \in Z$ consists of runs of at least n_k consecutive w_k separated by words of length less than $|w_k|$. If the sequence $\left(\frac{\log \log n_k}{\log |w_k|}\right)_{k \in \mathbb{N}}$ diverges to $+\infty$, then Z is not the $\mathbb{Z}$-projective subdynamics for any $\mathbb{Z}^2$ SFT X.

Corollary: There exist **effective Sturmian subshifts** which are not the one-dimensional projective subdynamics of any $\mathbb{Z}^2$ SFT. we also have a generalization of these two results to $\mathbb{Z}^d$ SFTs

Theorem [Quas-S]:

A larger class of **effective Sturmian subshifts** are not the one-dimensional projective subdynamics of any $\mathbb{Z}^2$ SFT.

There exist certain **Toeplitz subshifts** (almost odometers with fast growing periods) which are not the one-dimensional projective subdynamics of any $\mathbb{Z}^2$ SFT.

Open question: What about realizations of other zero-entropy $\mathbb{Z}$ shifts as projective subdynamics of $\mathbb{Z}^2$ ($\mathbb{Z}^d$) SFTs?