Multidimensional symbolic dynamics

(Minicourse – Lecture 3)

Michael H. Schraudner

Centro de Modelamiento Matemático
Universidad de Chile

mschraudner@dim.uchile.cl

www.cmm.uchile.cl/∼mschraudner

1st French-Chilean conference in Dynamics and Combinatorics

January 26th, 2012 – Cap Hornu, Baie de Somme

Subsystems of $\mathbb{Z}$ SFTs and $\mathbb{Z}$ sofics

Non-trivial mixing $\mathbb{Z}$ SFTs and $\mathbb{Z}$ sofics have a rich family of subsystems:

Theorem [Krieger's embedding theorem (1981)]:
Let X be a mixing $\mathbb{Z}$ SFT and let Y be a $\mathbb{Z}$ subshift such that $h_{\mathrm{top}}(X) > h_{\mathrm{top}}(Y)$ and $\mathrm{Per}(Y) \hookrightarrow \mathrm{Per}(X)$ then there exists an embedding $\phi : Y \hookrightarrow X$. marker-filler method

hence Y can be thought of as a subsystem of X and this result describes all symbolic subsystems

Moreover, given a proper subsystem $Z \subsetneq X$ of a mixing $\mathbb{Z}$ SFT X the embedding **can be chosen to be disjoint from the subsystem** Z, i.e. $\phi : Y \hookrightarrow X \setminus Z$.

Similar statement in the measurable setting by the **Jewett-Krieger theorem**: Every measurable $\mathbb{Z}$ system with finite entropy can be realized by a uniquely ergodic $\mathbb{Z}$ subshift.

Corollary: The same results hold for mixing $\mathbb{Z}$ sofics S.

S contains an increasing family of mixing $\mathbb{Z}$ SFTs $(X_n \subsetneq S)_{n\in\mathbb{N}}$ such that $h_{\mathrm{top}}(S) = \lim_{n\to\infty} h_{\mathrm{top}}(X_n)$

Positive entropy mixing $\mathbb{Z}$ SFTs and $\mathbb{Z}$ sofics are equally rich in subsystems; contain a tremendous collection of **pairwise disjoint** (minimal) subsystems with dense/nearly full entropies.

Subsystems of $\mathbb{Z}^d$ SFTs and $\mathbb{Z}^d$ sofics

Similar embedding result for a special subclass of $\mathbb{Z}^d$ SFTs:

Theorem [Lightwood's embedding theorem (2003)]:

Let X be a topologically mixing $\mathbb{Z}^2$ SFT having the **square-filling** property and let Y be a $\mathbb{Z}^2$ subshift **without periodic points** such that $h_{\mathrm{top}}(X) > h_{\mathrm{top}}(Y)$ then there exists an embedding $\phi: Y \hookrightarrow X$.

Let X be a $\mathbb{Z}^d$ SFT having the uniform filling property and containing a fixed point and let Y be a $\mathbb{Z}^d$ subshift **without periodic points** such that $h_{\mathrm{top}}(X) > h_{\mathrm{top}}(Y)$ then there exists an embedding $\phi: Y \hookrightarrow X$.

There is an analogous Jewett-Krieger theorem for $\mathbb{Z}^d$ shifts (Rosenthal) as well as several results on existence of completely positive entropy measures and Bernoulli measures in the presence of the uniform filling property (Robinson-Sahin).

 mschraudner@dim.uchile.cl

A rare completely general $\mathbb{Z}^d$ result:

Theorem [Desai (2006)]:

Any $\mathbb{Z}^d$ SFT X with $h_{\text{top}}(X) > 0$ contains a family of $\mathbb{Z}^d$ **subSFTs** with entropies dense in the interval $[0, h_{\text{top}}(X)]$.

Any $\mathbb{Z}^d$ sofic S with $h_{\text{top}}(X) > 0$ contains a family of $\mathbb{Z}^d$ **subsofics** with entropies dense in the interval $[0, h_{\text{top}}(S)]$.

Moreover every real number in the interval $[0, h_{\text{top}}(X)]$ resp. $[0, h_{\text{top}}(S)]$ is the entropy of some $\mathbb{Z}^d$ subshift subsystem of X resp. S.

nice, simple proof without actually computing the entropies, using colored grid plus excluding a large pattern

Hence again there are lots of subsystems.

Desai also proved that every $\mathbb{Z}^d$ sofic S has a $\mathbb{Z}^d$ SFT cover X which is ε-close in entropy, i.e. for every $\varepsilon > 0$ exists $\mathbb{Z}^d$ SFT X with $h_{\text{top}}(X) < h_{\text{top}}(S) + \varepsilon$ such that there is a factor map $\phi : X \twoheadrightarrow S$.

Open question (Weiss): Does every $\mathbb{Z}^d$ sofic S have an equal entropy $\mathbb{Z}^d$ SFT cover X such that $\phi : X \twoheadrightarrow S$ is an entropy preserving factor map?

True for $\mathbb{Z}$ sofics (Coven-Paul). Classes of entropies of $\mathbb{Z}^d$ SFTs and $\mathbb{Z}^d$ sofics coincide (Hochman-Meyerovitch).

Optimality of Desai's results – Non-separation of subsystems for $\mathbb{Z}^d$ sofics

Desai's result **does NOT guarantee disjointness** of subsystems nor the existence of a family of $\mathbb{Z}^d$ **subSFTs** with dense entropies for the case of $\mathbb{Z}^d$ sofics. In fact:

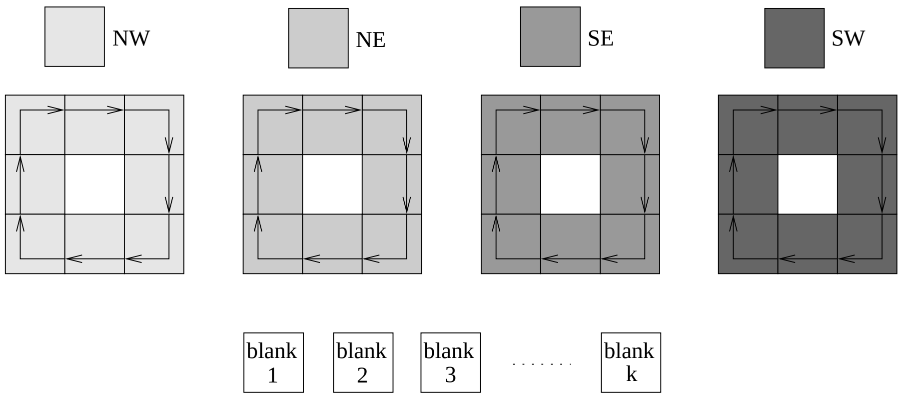
Theorem [Boyle-Pavlov-S]:
Suppose $d \geqslant 2$ and we are given $0 < M \in \mathbb{R}$, then there exists a $\mathbb{Z}^d$ sofic S with $h_{\mathrm{top}}(S) > M$ such that:

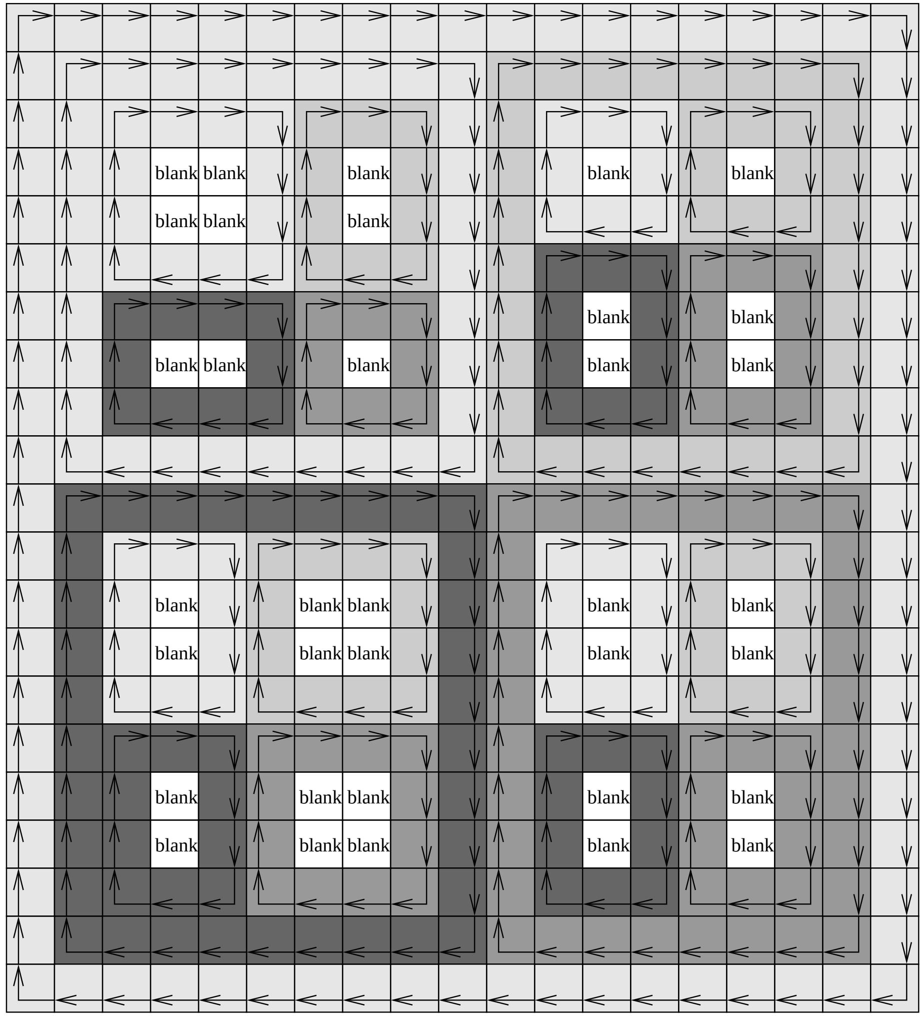
- S does contain a **unique minimal subsystem** being a fixed point. no other periodic points
- All subsystems of S have to contain this fixed point and thus can not be disjoint.
- S does contain a **unique** $\mathbb{Z}^d$ **subSFT**. again the unique fixed point
- S has an **equal entropy** $\mathbb{Z}^d$ SFT cover X with $\phi: X \twoheadrightarrow S$ (entropy preserving factor map) such that the ϕ-preimage of the unique fixed point in S is a zero entropy $\mathbb{Z}^d$ subSFT $K \subsetneq X$ containing all minimal subsystems and again **every subsystem of X has to intersect K**.

Moreover (at least) for $d = 2$ the $\mathbb{Z}^2$ sofic S and its $\mathbb{Z}^2$ SFT cover X can be chosen **topologically mixing** and of **topologically completely positive entropy**.

for $d > 2$ construction extends the technique of Hochman-Meyerovitch (2007) + example of Quas-Sahin + new ideas

The non-separation example in $\mathbb{Z}^2$





Factors of $\mathbb{Z}$ SFTs and $\mathbb{Z}$ sofics

Non-trivial mixing $\mathbb{Z}$ SFTs and $\mathbb{Z}$ sofics have a rich family of subshift factors:

Theorem [Boyle's lower entropy factor theorem (1983)]:
Let S be a mixing $\mathbb{Z}$ sofic and X a mixing $\mathbb{Z}$ SFT such that $h_{\mathrm{top}}(S) > h_{\mathrm{top}}(X)$. If $\mathrm{Per}(S) \downarrow \mathrm{Per}(X)$ then there exists a factor map $\phi : S \twoheadrightarrow X$. uses again Krieger's marker-filler method

Theorem [Equal entropy full shift factors, Marcus (1979)]:
Let $N \in \mathbb{N}$. If Y is a $\mathbb{Z}$ SFT with $h_{\mathrm{top}}(Y) \geqslant \log N$ then there exists a factor map from Y onto the full $\mathbb{Z}$-shift on N symbols.

Question: Can we obtain those (similar) factoring results also for the $\mathbb{Z}^d$ setting? open in general

Question [Madden-Johnson]: What about $\mathbb{Z}^d$ full shift factors?
In particular, given $N \in \mathbb{N}$ does every $\mathbb{Z}^d$ SFT X with $h_{\mathrm{top}}(X) \geqslant \log N$ factor onto the $\mathbb{Z}^d$ full shifts on N symbols? there we have some results

interesting to identify factors (=building blocks) of a given system

topological analogue (for $\mathbb{Z}^d$ SFTs) of Sinai's measurable factor theorem

Lower entropy full shift factors of $\mathbb{Z}^d$ SFTs and $\mathbb{Z}^d$ sofics

A negative lower entropy full shift factor result:

Examples of $\mathbb{Z}^d$ SFTs with non-separated subsystems yield an obstruction to the existence of factors with large number of disjoint subsystems.

Disjoint subsystems in a factor can only come from disjoint subsystems in the cover.

Theorem [Boyle-Pavlov-S]:
Suppose $d \geqslant 2$ and we are given $0 < M \in \mathbb{R}$, then there exists a $\mathbb{Z}^d$ SFT X with entropy $h_{\mathrm{top}}(X) > M$ which has

- no non-trivial block gluing $\mathbb{Z}^d$ **subshift** factor. block gluing subshifts have disjoint subsystems
- no non-trivial $\mathbb{Z}^d$ full shift factor.
- only $\mathbb{Z}^d$ subshift factors Y for which the orbit closure $Y_{\mathsf{Min}} \subseteq Y$ of its minimal subsystems has zero entropy $h_{\mathrm{top}}(Y_{\mathsf{Min}}) = 0$
- an equal entropy, proper $\mathbb{Z}^d$ sofic factor with topologically completely positive entropy.

Moreover (at least) for $d = 2$ the $\mathbb{Z}^2$ SFT X can be chosen **topologically mixing** and with **topologically completely positive entropy**. use the non-separation examples seen before

Similarly the $\mathbb{Z}^d$ sofics with non-separated subsystems immediately give the following negative factor results:

Theorem [Boyle-Pavlov-S]:
Suppose $d \geqslant 2$ and we are given $0 < M \in \mathbb{R}$, then there exists a $\mathbb{Z}^d$ sofic S of entropy $h_{\text{top}}(S) > M$ which has

- a unique $\mathbb{Z}^d$ SFT factor being the unique fixed point in S. The same holds for all factors of S.
- no non-trivial block gluing $\mathbb{Z}^d$ **subshift** factor.
- in particular no non-trivial full shift factor.
- an equal entropy $\mathbb{Z}^d$ sofic factor of topologically completely positive entropy.

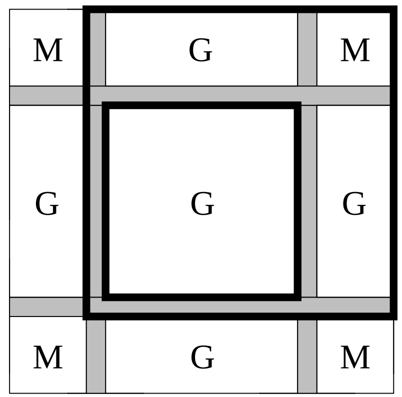
Question: Are there any (general) positive results?

A positive lower entropy full shift factor result:

Theorem [Boyle-Pavlov-S]:
Let $N \in \mathbb{N}$. Any block gluing $\mathbb{Z}^d$ **subshift** X with $h_{\text{top}}(X) > \log N$ factors onto

- the full $\mathbb{Z}^d$ shift on N symbols.
- a family of strongly irreducible $\mathbb{Z}^d$ SFTs with entropies dense in the interval $[0, h_{\text{top}}(X)]$.
- all lower entropy $\mathbb{Z}^d$ SFTs which have a safe symbol.

constructive proof using markers and a coding argument



Again these results emphasize the **importance of a uniform mixing condition**.

Definition: A $\mathbb{Z}^d$ subshift Y has a **safe symbol** if its alphabet contains an element $a \in \mathcal{A}$ such that for any point $y \in Y$ replacing the symbol at any coordinate in $\mathbb{Z}^d$ by the symbol a yields again a point in Y.

 Michael Schraudner mschraudner@dim.uchile.cl

Equal entropy full shift factors of $\mathbb{Z}^d$ SFTs

Conceptual problem: There are not a lot of $\mathbb{Z}^d$ SFT examples where the entropy is known to be $\log N$ with $N \in \mathbb{N}$.

One class of those $\mathbb{Z}^d$ SFTs are algebraic group shift over an alphabet being a finite group $\mathcal{A} = G$:

An **algebraic $\mathbb{Z}^d$ group shift** $X \leqslant G^{\mathbb{Z}^d}$ is a closed, shift-invariant subgroup.

Facts: $\mathbb{Z}^d$ algebraic group shifts

- are $\mathbb{Z}^d$ SFTs.
- have dense periodic points, thus in particular the undecidability questions vanish.
- have entropy $\log N$ for $N \in \mathbb{N}$ being the cardinality of a certain normal subgroup of G.

Theorem [Boyle-S]:
Let X be a $\mathbb{Z}^d$ algebraic group shift then there exists an entropy preserving factor map from X to the corresponding $\mathbb{Z}^d$ full shift. plus much stronger results for G abelian*

*The strongest, most developed $\mathbb{Z}^d$ theory exists for algebraic group shifts built over an alphabet which is a finite abelian group using Pontryagin duality principles and the structure of Noetherian modules. Work by Schmidt, Einsiedler, Ward, ...

First negative equal entropy full shift factor result:

Theorem [Boyle-S]:
Let $d \geqslant 2$ and $N \in \mathbb{N}$. There exist $\mathbb{Z}^d$ SFTs X with $h_{\text{top}}(X) = \log N$ which do **not** factor onto the full $\mathbb{Z}^d$ shift on N symbols.

proof again uses techniques of Hochman-Meyerovitch to produce a contradiction for measures of clopen sets

Question: Is this only an artifact of not having strong enough mixing?

Theorem [Pavlov-S]:
Let $d \geqslant 3$ and $N \in \mathbb{N}$. There exist block gluing $\mathbb{Z}^d$ SFTs X with $h_{\text{top}}(X) = \log N$ which do **not** factor onto the full $\mathbb{Z}^d$ shift on N symbols. probably extensible to the $d = 2$ case

construction uses upgradability of certain non-block gluing $\mathbb{Z}^d$ SFTs – produces non-entropy minimal block gluing examples – however does not work for UFP or strongly irreducible $\mathbb{Z}^d$ SFTs

Hence it seems the equal entropy full shift factor problem is more complicated.

compare with general equal entropy factor problem in $\mathbb{Z}$

Sketch of proof: Produce a measure-of-clopen-sets-obstruction.

For our obstruction pick a prime p that divides $N > 1$ and choose $K > N$ not divisible by p.

Start with the result of Hochman-Meyerovitch (2007):

The **upper frequency** of $\mathcal{A}' \subsetneq \mathcal{A}$ in a point $x \in X$ is defined to be

$$\limsup_{n\to\infty} \frac{1}{\operatorname{card} F(n)} \operatorname{card}\{\vec{\jmath} \in F(n) : x_{\vec{\jmath}} \in \mathcal{A}'\}$$

where $F(n) := \{\vec{\imath} \in \mathbb{Z}^d \mid \|\vec{\imath}\|_\infty \leqslant n\}$.

If the $\limsup$ is a limit, then it gives the **frequency** of $\mathcal{A}'$ in x.

Theorem [Hochman-Meyerovitch]: Suppose $r \in [0,1]$ is right recursively enumerable. Then there exists a zero entropy $\mathbb{Z}^d$ SFT Z and a subset $\mathcal{A}' \subsetneq \mathcal{A}$ of the alphabet of Z such that:

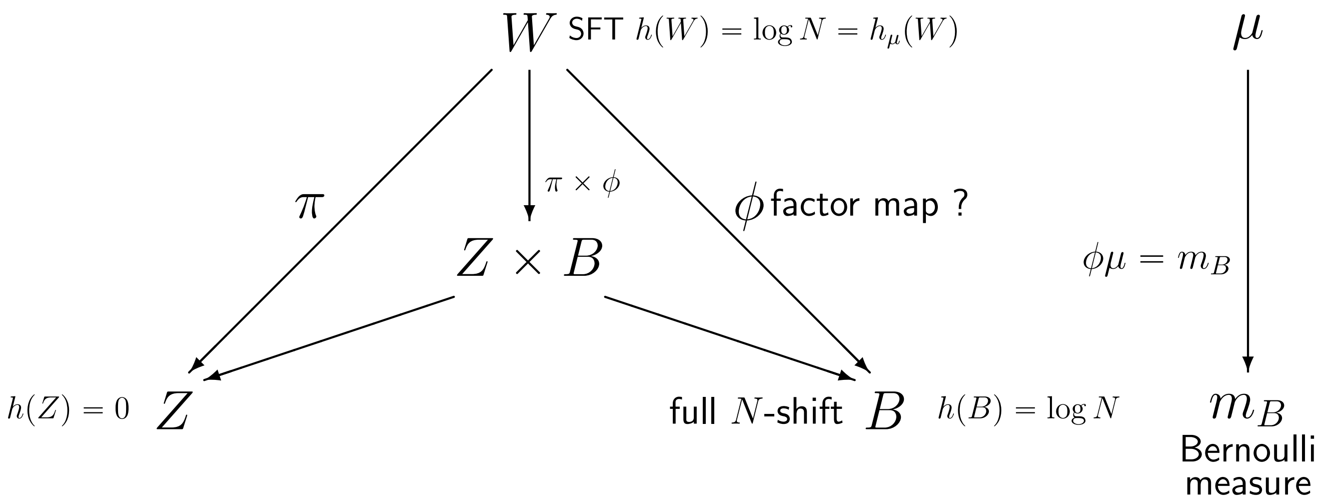
- for any point $z \in Z$, the upper frequency of $\mathcal{A}'$ is at most r.
- there exists a point of Z in which $\mathcal{A}'$ has frequency r.

Take $r = (\log N)/(\log K)$ for our example (right recursively enumerable by $\log$ power series).

Now replace the symbols of $\mathcal{A}'$ in every point $z \in Z$ independently with one of K copies. Define

$$\widetilde{\mathcal{A}} = (\mathcal{A} \backslash \mathcal{A}') \;\cup\; \{(a,i) : a \in \mathcal{A}' \wedge i \in \{1,\ldots,K\}\}$$

Define the subshift W consisting of all configurations $w \in \widetilde{\mathcal{A}}^{\mathbb{Z}^d}$ such that the one-block code $\pi : \widetilde{\mathcal{A}} \to \mathcal{A}$ given by $a \mapsto a$ if $a \notin \mathcal{A}'$ and $(a,i) \mapsto a$ if $a \in \mathcal{A}'$ sends W onto Z.



Given μ on W we denote by $\{\mu_z\}$ the ν-a.e. unique family of Borel probabilities on the fibers $\pi^{-1}z$ such that $\mu(E) = \int \mu_z(E \cap \pi^{-1}z)\, d\nu(z)$, for all Borel sets E.

Given $\nu = \pi\mu$ on Z, let $\widetilde{\nu}$ be the unique lift of ν such that $\widetilde{\nu}_z = \beta_z$ for ν-a.e. z.

Lemma: Suppose Z is a $\mathbb{Z}^d$ subshift; W, π and $\widetilde{\nu}$ as above; $\mu \in \mathcal{M}(W)$; and $\pi\mu = \nu$. Then

$$h_\mu(W) \leqslant h_\nu(Z) + \nu\big(\textstyle\bigcup_{a\in\mathcal{A}'}[a]_0\big)\log K$$

with equality holding if and only if $\mu = \widetilde{\nu}$.

By the Lemma and the variational principle, we have $h(W) = \log N$ and $\mu = \widetilde{\nu}$.

Using the disjointness of zero entropy and Bernoulli we get for ν-almost all points $z \in Z$ that the factor map $\phi|_{\pi^{-1}z}$ maps $\widetilde{\nu}_z$ to m_B.

Pick such a $z \in Z$.

$$p \,|\, N \implies \exists\, C \subset B \text{ clopen: } m_B(C) = \tfrac{1}{p}$$

$$\phi \text{ continuous} \implies D = (\phi|_{\pi^{-1}z})^{-1}(C) \subset \pi^{-1}z \text{ clopen}$$

But then $\widetilde{\nu}_z(D) = m_B(C) = \frac{1}{p}$.

As $p \mid K$ there is no clopen set $D \subset \pi^{-1}z$ of $\widetilde{\nu}_z$-measure $\frac{1}{p}$.

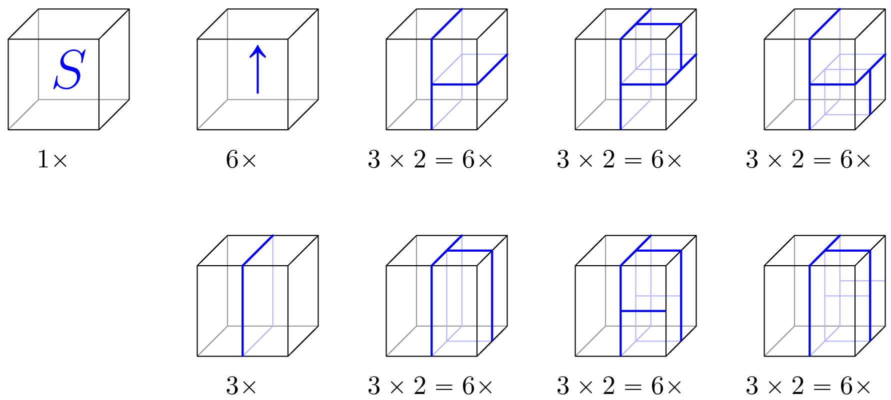
There is no factor map ϕ from W onto B.

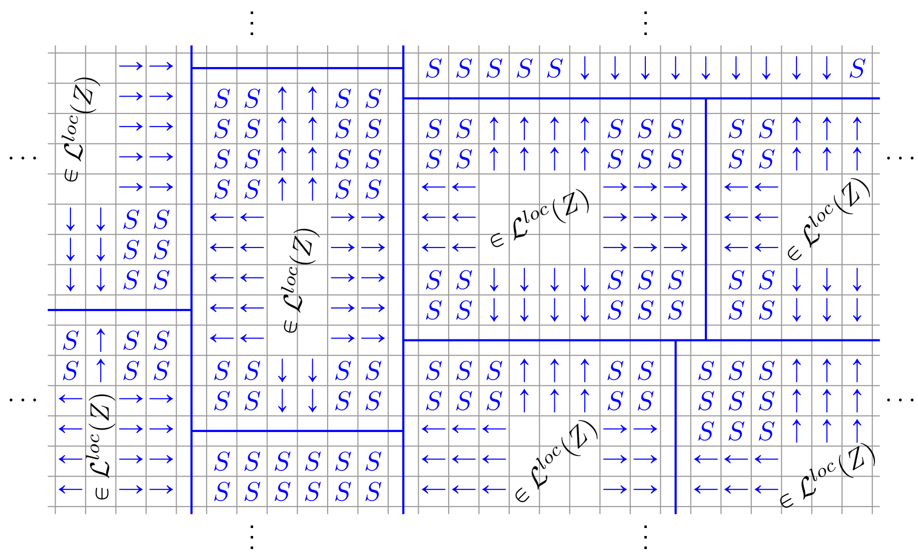
Upgradeable $\mathbb{Z}^d$ SFTs – Making things block gluing

Definition: A $\mathbb{Z}^3$ shift of finite type X is **upgradable** if there is a non-negative real constant $C \in \mathbb{R}_0^+$ so that $\left|\mathcal{L}^{loc}_{k,l,m}(X)\right| \leqslant e^{h_{\text{top}}(X)klm + C(kl+km+lm)}$ for all $k, l, m \in \mathbb{N}$.
(quiet restrictive condition, satisfied e.g. for $\mathbb{Z}^d$ full shifts)

Theorem: For any upgradable $\mathbb{Z}^3$ SFT Z with $h_{\text{top}}(Z) > 0$, there exists a block gluing $\mathbb{Z}^3$ SFT $\widetilde{Z}$ containing Z as a subsystem with $h_{\text{top}}(\widetilde{Z}) = h_{\text{top}}(Z)$.

Idea: Introduce new (wall) symbols to generate cuboid cells in which only locally admissible patterns of $\widetilde{Z}$ are allowed. (elaborate $\mathbb{Z}^3$ version of wire shift technique)





 Michael Schraudner mschraudner@dim.uchile.cl

Measures of maximal entropy

In mixing $\mathbb{Z}$ SFTs: Unique measure of maximal entropy (Parry measure).

Very explicit control, Markov measure given by the matrix, Gibbs property, exponential decay.

In $\mathbb{Z}^d$ SFTs:

Measures of maximal entropy are still Markov (random fields), but it is hard to get them.

First examples of $\mathbb{Z}^2$ SFTs with **multiple measures of maximal measures**: Burton-Steif's iceberg model.

strongly irreducible, so mixing does not help here, full support, symmetry of the alphabet and rules

Criteria for uniqueness exist (Markley-Paul, van den Berg-Maes, Haeggstroem, Pavlov), but are not invariant

Big open question!

Summary of this minicourse

- Definitions and questions of $\mathbb{Z}$ symbolic dynamics can be generalized naturally to the $\mathbb{Z}^d$ framework.
- However the answers and results are very different from the classical theory – properties generalize if at all only to certain subclasses.
- The useful structures of $\mathbb{Z}$ SFTs – matrices, graphs, algebraic invariants – are not accessible in $\mathbb{Z}^d$.
- The world of multidimensional $\mathbb{Z}^d$ SFTs $(d > 1)$ is more varied, vastly richer and much more complicated than the class of $\mathbb{Z}$ SFTs.
- There is no global theory for the very inhomogeneous class of $\mathbb{Z}^d$ SFTs, but many results for certain subclasses.
- Undecidability and recursion theory plays a large role – strongest construction techniques rely on Turing machines.
- Uniform mixing conditions help in avoiding pathologies and undecidability issues.
- We are still exploring the "landscape", finding interesting examples, discovering new properties and unexpected phenomena.

A lot of progress over the last 10 years, but there are still many open questions!

Michael Schraudner mschraudner@dim.uchile.cl