$\mathbb{Z}^d$ SHIFTS OF FINITE TYPE WITHOUT EQUAL ENTROPY FULL SHIFT FACTORS

MIKE BOYLE AND MICHAEL SCHRAUDNER

ABSTRACT. For $d \geq 2$, we use results of Hochman and Meyerovitch to construct examples of $\mathbb{Z}^d$ shifts of finite type of entropy $\log N$, $N \in \mathbb{N}$, which cannot factor topologically onto the $\mathbb{Z}^d$ Bernoulli shift on N symbols.

1. INTRODUCTION

For $d > 1$, the $\mathbb{Z}^d$ shifts of finite type (SFTs) are a drastically more problematic and diverse class of systems than the $\mathbb{Z}$ SFTs [16, 17, 18]. Most distinguishing features of $\mathbb{Z}$ SFTs generalize, if at all, only to a subclass of $\mathbb{Z}^d$ SFTs when $d > 1$ (for a rare exception, see [5]). As a part of the effort to understand the $\mathbb{Z}^d$ SFTs, we consider below the following problem studied by Johnson and Madden:

Question 1.1. [9] If N is a positive integer and X is a $\mathbb{Z}^d$ SFT with entropy $h(X) \geq \log N$, must there exist a continuous factor map from X onto the full $\mathbb{Z}^d$ shift on N symbols?

Here a factor map between two $\mathbb{Z}^d$ subshifts means a surjective map that intertwines the shift actions. A factor map is called topological if it is in addition continuous and it is called measurable if the factor map is a measure preserving Borel map.

Question 1.1 is interesting as a matter of basic symbolic dynamics and also as a question of whether there is a topological analogue (for SFTs) to one of the fundamental theorems in ergodic theory, Sinai's Factor Theorem [19, 20]: an ergodic measure preserving transformation on Lebesgue probability space has as a measurable factor every Bernoulli shift of equal or smaller entropy. Sinai proved his theorem for $\mathbb{Z}$ actions; his theorem was generalized to actions by discrete amenable groups ([15]; alternately, see [10]), in particular actions by $\mathbb{Z}^d$.

In the case $d = 1$, the answer to Question 1.1 is emphatically yes; the proofs are quite different for the cases of equal entropy [2, 13] and unequal entropy [3]. For $d > 1$ and $h(X) > \log N$, the answer to Question 1.1 is not known, but in this case it is known that the answer is yes if in addition X has the mixing property of being "corner gluing" [6, 9].

In this note, we will answer Question 1.1 in the equal-entropy case: given integers $N, d > 1$, we will construct a $\mathbb{Z}^d$ SFT with entropy $\log N$ which has no continuous factor map onto the full $\mathbb{Z}^d$ shift on N symbols. It should not be surprising that

such examples exist; on the other hand, it appears to be nontrivial to actually construct and verify an example.

We are assuming a basic familiarity with symbolic dynamics. Recall in particular that the topological $\mathbb{Z}^d$ entropy of a $\mathbb{Z}^d$ subshift X is by definition

$$h(X) = h_d(X) = \limsup_n \frac{1}{\operatorname{card} F(n)} \log \operatorname{card}\{x|_{F(n)} : x \in X\}$$

where $F(n) = \{\mathbf{v} \in \mathbb{Z}^d : 0 \le v_i < n, 1 \le i \le d\}$. The measure-theoretic entropy $h_\mu(X)$ of X with respect to a shift-invariant Borel probability μ is generalized analogously from the $\mathbb{Z}$ case.

We thank our ergodic guru Dan Rudolph for references to the literature and for the proof of Lemma 2.12.

2. $\mathbb{Z}^d$ SHIFTS OF FINITE TYPE WITHOUT EQUAL-ENTROPY BERNOULLI FACTORS

The purpose of this section is to prove the following result.

Theorem 2.1. *Suppose N and d are positive integers strictly greater than 1. Then there exists a $\mathbb{Z}^d$ shift of finite type W of topological $\mathbb{Z}^d$ entropy $\log N$ but for which there is not a continuous shift commuting epimorphism onto the full $\mathbb{Z}^d$ shift on N symbols.*

Given d we let $F(n)$ be the subset $\{\mathbf{v} \in \mathbb{Z}^d : 0 \le v_i < n, 1 \le i \le d\}$ of $\mathbb{Z}^d$. For a point x in a $\mathbb{Z}^d$ subshift and a set $\mathcal{A}$ of symbols, the *upper frequency* of $\mathcal{A}$ in x is defined to be $\limsup_n (1/\operatorname{card} F(n)) \operatorname{card}\{\mathbf{v} \in F(n) : x(\mathbf{v}) \in \mathcal{A}\}$. If the $\limsup$ here is a limit, then it is the *frequency* of $\mathcal{A}$ in x. A real number r is *right recursively enumerable* if there exists a sequence of rational numbers $r(n) \ge r$ converging to r and a Turing machine which given $n \in \mathbb{N}$ produces output $r(n)$. Hochman and Meyerovitch proved that for $d > 1$ the set of entropies of $\mathbb{Z}^d$ SFTs is exactly the set of nonnegative right recursively enumerable real numbers. The following result is the main part of their proof, and the main ingredient for our proof of Theorem 2.1.

Theorem 2.2. [8] *Suppose r is a right recursively enumerable real number and $0 \le r \le 1$. Then there is a zero entropy $\mathbb{Z}^d$ SFT Z and a subset $\mathcal{A}$ of the alphabet of Z such that the following hold:*

- *for any point z in Z, the upper frequency of $\mathcal{A}$ is at most r*
- *there exists a point of Z in which $\mathcal{A}$ has frequency r.*

(In addition, when r above is computable, the construction of Z was elaborated in [8] so that $\mathcal{A}$ has frequency r in every point of Z.) In [8] a construction for Z in Theorem 2.2 is given in the case $d = 2$, with Z a minimal SFT built over a tiling subshift by a theorem of Mozes [14]. As remarked in [8], given $d \ge 1$ and a $\mathbb{Z}^d$ SFT X with alphabet Σ, one can form a $\mathbb{Z}^{d+1}$ SFT

$$X' = \{x' \in \Sigma^{\mathbb{Z}^{d+1}} : \forall j \in \mathbb{Z} \ \exists x \in X \ \forall \mathbf{u} \in \mathbb{Z}^d \ x'(\mathbf{u}, j) = x(\mathbf{u})\}$$

and here the topological $\mathbb{Z}^{d+1}$ entropy of X' will equal the topological $\mathbb{Z}^d$ entropy of X. Likewise, the frequency properties of X with respect to $\mathcal{A} \subset \Sigma$ needed in Theorem 2.2 will carry forward.

We introduce a little notation for another ingredient to our proof (elaborating a part of [8]). Suppose Z is a $\mathbb{Z}^d$ subshift, $\Sigma(Z)$ is its alphabet, $\mathcal{A} \subset \Sigma(Z)$ and $K \in \mathbb{N}$. Define

$$\mathcal{A}' \quad = \quad (\Sigma(Z) \setminus \mathcal{A}) \ \cup \ \{(a, i) : a \in \mathcal{A}, i \in \{1, \dots, K\}\} \ .$$

Define $\pi : \mathcal{A}' \to \Sigma(Z)$ by $a \mapsto a$ if $a \notin \mathcal{A}$ and $(a, i) \mapsto a$ if $a \in \mathcal{A}$. Define $W(Z, K, \mathcal{A})$ to be the subshift W consisting of all configurations w on alphabet $\mathcal{A}'$ such that the one-block code defined by π sends w into Z. We also use π to name this code $W \to Z$. Given a measure μ on a subshift and a set of symbols $\mathcal{E}$ from its alphabet, define $\mu(\mathcal{E}) = \mu\{x : x(\mathbf{0}) \in \mathcal{E}\}$, and for brevity write $\mu(\{a\}) = \mu(a)$. Given $\mathcal{A}$ above, and $z \in Z$ with some $z(\mathbf{v}) \in \mathcal{A}$, define

$$\gamma_z : \pi^{-1}z \to \prod_{\mathbf{v} \in \mathbb{Z}^d : \ z(\mathbf{v}) \in \mathcal{A}} \{1, \dots, K\}$$

by setting $(\gamma_z(w))(\mathbf{v}) = i$ when $w(\mathbf{v}) = (a, i)$. We say that a measure on $\pi^{-1}z$ is uniform measure, denoted β_z, if it is the pullback by γ_z to $\pi^{-1}z$ of the measure on the image of γ_z which is the product measure of uniform discrete measure on $\{1, \dots, K\}$. (If $\pi^{-1}z$ is a single point, then we define β_z to give this point measure 1.)

Given μ on W we let $\{\mu_z\}$ be the ν-a.e. unique family of Borel probabilities on the fibers $\pi^{-1}z$ such that $\mu(E) = \int \mu_z(E \cap \pi^{-1}z) \, d\nu(z)$, for all Borel sets E. Given ν on Z, let $\widetilde{\nu}$ be the unique lift of ν in $\mathcal{M}(W)$ such that $\widetilde{\nu}_z = \beta_z$ for ν-a.e. z. Here and below $\mathcal{M}(W)$ denotes the set of shift invariant Borel probabilities of some subshift W.

The next lemma is intuitively clear from consideration of the $\mathbb{Z}^d$ ergodic theorem [12]. We give an elementary proof.

Lemma 2.3. *Suppose Z is a $\mathbb{Z}^d$ subshift; $W = W(Z, K, \mathcal{A})$, π and $\widetilde{\nu}$ are defined as above; $\mu \in \mathcal{M}(W)$; and $\pi\mu = \nu$. Then*

$$h_\mu(W) \ \leq \ h_\nu(Z) + \nu(\mathcal{A}) \log K$$

with equality holding if and only if $\mu = \widetilde{\nu}$.

Proof. Given finite measurable partitions $\mathcal{P}, \mathcal{Q}$ of W, recall the conditional entropy formula

(2.4) $$H_\mu(\mathcal{P}|\mathcal{Q}) = \sum_B (\mu B) H_{\mu_B}(\mathcal{P}_B)$$

(2.5) $$\leq \sum_B (\mu B) \log(\mathrm{card}(\mathcal{P}_B))$$

where the sum is over the elements B of $\mathcal{Q}$ with $\mu B > 0$, $\mathcal{P}_B$ is the partition $B \cap \mathcal{P}$ of B and μ_B is the conditional probability on B defined from μ.

Given a finite subset F of $\mathbb{Z}^d$, let $\mathcal{P}^F$ be the partition of W according to the configurations $w|_F$, and let $\mathcal{Q}^F$ be the partition of W according to the configurations $(\pi w)|_F$. For F, G disjoint finite subsets of $\mathbb{Z}^d$, we recall a subadditivity relation for conditional entropy:

(2.6) $$H_\mu(\mathcal{P}^{F \cup G}|\mathcal{Q}^{F \cup G}) = H_\mu(\mathcal{P}^F|\mathcal{Q}^{F \cup G}) + H_\mu(\mathcal{P}^G|\mathcal{Q}^{F \cup G} \vee \mathcal{P}^F)$$

(2.7) $$\leq H_\mu(\mathcal{P}^F|\mathcal{Q}^F) + H_\mu(\mathcal{P}^G|\mathcal{Q}^G) \ .$$

Setting $F(n) = \{\mathbf{v} \in \mathbb{Z}^d : 0 \le v_i < n, 1 \le i \le d\}$, we have

$$(2.8) \qquad h_\mu(W) = \lim_n \frac{1}{\operatorname{card} F(n)} \big(H_\mu(\mathcal{Q}^{F(n)}) + H_\mu(\mathcal{P}^{F(n)}|\mathcal{Q}^{F(n)})\big)$$

$$(2.9) \qquad h_\mu(W) - h_\nu(Z) = \lim_n \frac{1}{\operatorname{card} F(n)} H_\mu(\mathcal{P}^{F(n)}|\mathcal{Q}^{F(n)})$$

$$(2.10) \qquad h_\mu(W) - h_\nu(Z) = \inf_n \frac{1}{\operatorname{card} F(n)} H_\mu(\mathcal{P}^{F(n)}|\mathcal{Q}^{F(n)})$$

where (2.10) follows from the subadditivity relation (2.7).

Now we consider the measure $\widetilde{\nu}$. By direct computation, we have

$$(2.11) \qquad H_{\widetilde{\nu}}(\mathcal{P}^{F(1)}|\mathcal{Q}^{F(1)}) = \nu(\mathcal{A}) \log K .$$

Suppose F, G are disjoint finite subsets of $\mathbb{Z}^d$; $\mathcal{P} = \mathcal{P}^F$; $\mathcal{Q}$ is $\mathcal{Q}^F$ or $\mathcal{Q}^{F \cup G}$; B is an element of $\mathcal{Q}$; $\widetilde{\nu}B > 0$; and $N_B = \operatorname{card}\{\mathbf{v} \in F(n) : (\pi w)_\mathbf{v} \in \mathcal{A}$ for a.a. $w \in B\}$. Then $H_{\widetilde{\nu}_B}(\mathcal{P}_B) = N_B \log K$. Consequently,

$$H_{\widetilde{\nu}}(\mathcal{P}^F|\mathcal{Q}^{F \cup G}) = H_{\widetilde{\nu}}(\mathcal{P}^F|\mathcal{Q}^F) .$$

Similarly,

$$H_{\widetilde{\nu}}(\mathcal{P}^G|\mathcal{Q}^{F \cup G}) = H_{\widetilde{\nu}}(\mathcal{P}^G|\mathcal{Q}^{F \cup G} \vee \mathcal{P}^F) .$$

Therefore for $\mu = \widetilde{\nu}$, the inequality (2.7) is equality. By an induction we obtain for every n that

$$H_{\widetilde{\nu}}(\mathcal{P}^{F(n)}|\mathcal{Q}^{F(n)}) = (\operatorname{card} F(n)) H_{\widetilde{\nu}}(\mathcal{P}^{F(1)}|\mathcal{Q}^{F(1)})$$

and then from (2.9) and (2.11) we conclude that for all n,

$$h_{\widetilde{\nu}}(W) - h_\nu(Z) = \frac{1}{\operatorname{card} F(n)} H_{\widetilde{\nu}}(\mathcal{P}^{F(n)}|\mathcal{Q}^{F(n)}) = \nu(\mathcal{A}) \log K .$$

On the other hand, if $\mu \ne \widetilde{\nu}$, then for some n and some $B \in \mathcal{Q}^{F(n)}$ with $\mu B > 0$, there will be elements of $B \cap \mathcal{P}^{F(n)}$ with distinct measures, so that with $F = G = F(n)$ the inequality (2.7) becomes strict, and therefore

$$\frac{1}{\operatorname{card} F(n)} H_\mu(\mathcal{P}^{F(n)}|\mathcal{Q}^{F(n)}) < \frac{1}{\operatorname{card} F(n)} H_{\widetilde{\nu}}(\mathcal{P}^{F(n)}|\mathcal{Q}^{F(n)}) = h_{\widetilde{\nu}}(W) - h_\nu(Z) .$$

It then follows from (2.10) that $h_\mu(W) < h_{\widetilde{\nu}}(W)$. $\qquad\square$

The next lemma is an instance of a much more general disjointness theorem [7]; we thank Dan Rudolph for the simple proof below for the special case we use.

Lemma 2.12. *For $i = 1, 2, 3$ let $(X_i, \mathfrak{B}_i, \mu_i, \alpha_i)$ denote a μ_i-preserving $\mathbb{Z}^d$ action α_i on a Lebesgue probability space $(X_i, \mathfrak{B}_i)$. Suppose α_1 is zero entropy, α_2 is Bernoulli (i.e. μ_2 is the product measure of a measure on the coordinate alphabet) and there are measure preserving factor maps p_i from $(X_3, \mathfrak{B}_3, \mu_3, \alpha_3)$ to $(X_i, \mathfrak{B}_i, \mu_i, \alpha_i)$, $i = 1, 2$. Then $p_1 \times p_2$ defines a measure preserving factor map from $(X_3, \mathfrak{B}_3, \mu_3, \alpha_3)$ to the product system $(X_1 \times X_2, \mathfrak{B}_1 \times \mathfrak{B}_2, \mu_1 \times \mu_2, \alpha_1 \times \alpha_2)$.*

Proof. The map $p_1 \times p_2$ sends μ_3 onto a joining ν of μ_1 and μ_2 (a measure which is sent by the respective coordinate projections onto μ_1 and μ_2).

Write x in $X = X_1 \times X_2$ as (x_1, x_2). Let $\mathcal{P}$ and $\mathcal{Q}$ be the partitions of X according respectively to the symbols $x_1(\mathbf{0})$ and $x_2(\mathbf{0})$. Because α_1 has zero entropy, we have from $h_\nu(\alpha_1 \times \alpha_2) = h_{\mu_2}(\alpha_2)$ that for all n,

$$\inf_n \frac{1}{\operatorname{card} F(n)} H_\nu(\mathcal{Q}^{F(n)}) = \inf_n \frac{1}{\operatorname{card} F(n)} H_\nu(\mathcal{Q}^{F(n)}|\mathcal{P}^{F(n)}) .$$

Because the Bernoulli is i.i.d., the last equality implies that the partitions $\mathcal{P}^{F(n)}$ and $\mathcal{Q}^{F(n)}$ are independent for all n. This means that ν is the product measure $\mu_1 \times \mu_2$. $\qquad\square$

We can now prove Theorem 2.1.

Proof of Theorem 2.1. Pick a prime p dividing N and then an integer $K > N$ such that p does not divide K. Let $r = (\log N)/(\log K)$; by virtue of the power series for log, this r is recursively enumerable and therefore right recursively enumerable [8]. Appealing to Theorem 2.2, let Z be a zero entropy $\mathbb{Z}^d$ SFT with a set $\mathcal{A}$ of symbols occuring with frequency r. Define $W = W(Z, K, \mathcal{A})$ as earlier. By Lemma 2.3 and the variational principle, we have $h(W) = \log N$.

For a contradiction, let B denote the full $\mathbb{Z}^d$ shift on N symbols, and suppose f is a block code mapping W onto B. Let m_B be the uniform product measure on B, which is the unique measure of maximal entropy on B. There exists $\mu \in \mathcal{M}(W)$ such that $f\mu = m_B$, and $h_\mu(W) = \log N = h(W)$. Let ν be the image of μ in $\mathcal{M}(Z)$. Because $h_\mu(W) = h(W)$, it follows from Lemma 2.3 that $\mu = \widetilde{\nu}$.

It follows from Lemma 2.12 that for ν-almost-all z in Z, the restriction f_z of f to $\pi^{-1}z$ maps the measure $\widetilde{\nu}_z = \beta_z$ to m_B. Pick such a z. The map f is continuous from the Cantor set $\pi^{-1}z$ to B. Therefore, if C is a clopen set in B, the set $f_z^{-1}(C)$ will be clopen in $\pi^{-1}z$, with $\beta_z(f_z^{-1}C) = m_B(C)$. Because p divides N, there is a clopen set C in B such that $m_B(C) = 1/p$; because p does not divide K, there can be no clopen set D in $\pi^{-1}z$ such that $\beta_z(D) = 1/p$. This gives the contradiction. $\qquad\square$

Question 2.13. Our proof relied on the construction of Hochman and Meyerovitch, but we do not know how much this is an artifact of the proof. For example, suppose that $W = W(Z, K, \mathcal{A})$ is constructed from a nontrivial uniquely ergodic $\mathbb{Z}^2$ SFT Z with $\nu(\mathcal{A}) = 1/2$ and $K = 4$. Then $h(W) = \log 2$ and there is no longer a measures-of-clopen-sets obstruction. Must W factor onto the $\mathbb{Z}^d$ 2-shift?

Remark 2.14. An expansive action of $\mathbb{Z}^d$ by continuous automorphisms on a compact zero dimensional group is topologically conjugate to a $\mathbb{Z}^d$ SFT. Such SFTs are well known to have entropies of the form $\log N$, $N \in \mathbb{N}$. All of them factor onto Bernoulli shifts of equal entropy [4].

Question 2.15. Outside of algebraic systems, we do not know many examples "in nature" of $\mathbb{Z}^d$ SFTs for which the entropy is known and is equal to $\log N$ for some $N \in \mathbb{N}$. However, let X be the $\mathbb{Z}^2$ SFT whose configurations are all colorings of an infinite chessboard with three colors such that adjacent squares are colored differently. It follows [18, Example 3.2(2)] from a famous computation of Lieb [11] that $h(X) = (1/2)\log(64/27)$. Let Y be the product of $X \times X$ with the $\mathbb{Z}^2$ Bernoulli shift on 27 symbols; Y is a $\mathbb{Z}^2$ SFT of entropy $\log 64$. Does Y factor onto the $\mathbb{Z}^2$ Bernoulli shift on 64 symbols?

References

[1] R. L. Adler, D. Coppersmith and M. Hassner. *Algorithms for sliding block codes. An application of symbolic dynamics to information theory*, IEEE Trans. Inform. Theory **29** (1983), no. 1, 5–22.

[2] J. Ashley. *An extension theorem for closing maps of shifts of finite type*, Trans. Amer. Math. Soc. **336** (1993), no. 1, 389–420.

[3] M. Boyle. *Lower entropy factors of sofic systems*, Ergodic Theory Dynam. Systems **3** (1983), no. 4, 541–557.

[4] M. Boyle and M. Schraudner. $\mathbb{Z}^d$ *group shifts and Bernoulli factors*, preprint (2007).

[5] A. Desai, *Subsystem entropy for* $\mathbb{Z}^d$ *sofic shifts*, Indagationes Mathematicae **17** (2006), No. 3, 353-360.

[6] A. Desai, *A class of* $\mathbb{Z}^d$ *SFTs which factor onto lower entropy full shifts*, Proc. AMS, to appear.

[7] E. Glasner, J.-P. Thouvenot and B. Weiss. *Entropy theory without a past*, Ergodic Theory Dynam. Systems **20** (2000), no. 5, 1355–1370.

[8] M. Hochman and T. Meyerovitch. *A characterization of the entropies of multidimensional shifts of finite type*, Mathematics ArXiv DS/0703206 (2007).

[9] A. Johnson and K. Madden. *Factoring higher-dimensional shifts of finite type onto the full shift*, Ergodic Theory Dynam. Systems **25** (2005), no. 3, 811–822.

[10] J. W. Kammeyer and D. J. Rudolph, *Restricted orbit equivalence for actions of discrete amenable groups*, Cambridge Tracts in Mathematics **146**, Cambridge University Press, Cambridge, 2002.

[11] E. Lieb. *Residual entropy of square ice*, Phys. Rev. **162** (1967), 162-172.

[12] E. Lindenstrauss. *Pointwise theorems for amenable groups*, Inventiones Math. **146** (2001), no. 2, 259–295.

[13] B. Marcus. *Factors and extensions of full shifts*, Monatsh. Math. **88** (1979), no. 3, 239–247.

[14] S. Mozes. *Tilings, substitution systems and dynamical systems generated by them*, J. Analyse Math. **53** (1989), 139–186.

[15] D. Ornstein and B. Weiss. *Entropy and isomorphism theorems for actions of amenable groups*, J. Analyse Math. **48** (1987), 1–141.

[16] A. Quas and A. Sahin. *Entropy gaps and locally maximal entropy in* $\mathbb{Z}^d$ *subshifts*, Ergodic Theory Dynam. Systems **23** (2003), 1227-1245.

[17] Raphael M. Robinson. *Undecidability and nonperiodicity of tilings of the plane*, Inventiones Math. **12** (1971), 177-209.

[18] K. Schmidt. *Algebraic* $\mathbb{Z}^d$ *actions*, Lecture notes of PIMS Distinguished Chair Lectures, Pacific Institute for the Mathematical Sciences (2002), 61 pages.

[19] Ja. G. Sinai. *A weak isomorphism of transformations with invariant measure* (Russian), Dokl. Akad. Nauk SSSR **147** (1962), 797–800.

[20] Ja. G. Sinai. *On a weak isomorphism of transformations with invariant measure* (Russian), Mat. Sb. (N.S.) **63** (1964), 23–42.

MIKE BOYLE, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD 20742-4015, U.S.A., AND CENTRO DE MODELAMIENTO MATEMATICO, UNIVERSIDAD DE CHILE, AV. BLANCO ENCALADA 2120, SANTIAGO DE CHILE

E-mail address: mmb@math.umd.edu

URL: www.math.umd.edu/∼mmb

MICHAEL SCHRAUDNER, CENTRO DE MODELAMIENTO MATEMATICO, UNIVERSIDAD DE CHILE, AV. BLANCO ENCALADA 2120 PISO 7, SANTIAGO DE CHILE

E-mail address: mschraudner@dim.uchile.cl

URL: www.math.uni-heidelberg.de/studinfo/schraudner