ONE-DIMENSIONAL PROJECTIVE SUBDYNAMICS OF UNIFORMLY MIXING $\mathbb{Z}^d$ SHIFTS OF FINITE TYPE

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Abstract. We investigate under which circumstances the projective subdynamics of multidimensional shifts of finite type can be non-sofic. In particular we give a sufficient condition ensuring the one-dimensional projective subdynamics of such $\mathbb{Z}^d$ systems to be sofic and we show that this condition is already met (along certain resp. all sublattices) by most of the commonly used uniform mixing conditions. (Examples of the different situations are given.)

Complementary to this we are able to prove a characterization of one-dimensional projective subdynamics for strongly irreducible $\mathbb{Z}^d$ shifts of finite type for every $d \geq 2$: In this setting the class of possible subdynamics coincides exactly with the class of mixing $\mathbb{Z}$ sofics. This stands in stark contrast to the much more diverse situation in merely topologically mixing multidimensional shifts of finite type.

1. Introduction

This paper continues the study of projective subdynamics of multidimensional shifts of finite type started in a previous joint work [13] by Ronnie Pavlov and the author. Here we focus on a new aspect, namely demanding some form of mixing.

Given any $\mathbb{Z}^d$ subshift X ($d \geq 2$), define lower-dimensional subshifts by projecting points in X onto sublattices of $\mathbb{Z}^d$. Those subshifts – called the projective subdynamics of X – still contain valuable information about the dynamics of X. While general properties of higher-dimensional shifts are hard to study even in the case of X being a $\mathbb{Z}^d$ shift of finite type (SFT) its one-dimensional projective subdynamics are mostly accessible. Investigating them does provide insights leading to a better understanding of the symbolic $\mathbb{Z}^d$ system they build up to.

In the earlier paper [13] the question of which subshifts actually appear as projective subdynamics of general $\mathbb{Z}^d$ SFTs has been investigated without imposing any additional assumptions like a mixing property. A focus was put on the class of $\mathbb{Z}$ sofic shifts for which a complete and explicit classification was given. However it was also shown that the local rules used to define a $\mathbb{Z}^d$ SFT in general are far from forcing local or even nearly local rules for its lower-dimensional projective subdynamics. In fact a decrease of dimension by just one already provides a lot of freedom in using those $\mathbb{Z}^d$ local rules which turns out powerful enough to create rather exotic and highly non-sofic lower-dimensional projective subdynamics. E.g. arbitrary limit sets of cellular automata and a large class of $\mathbb{Z}$ coded systems are easily realizable as long as we accept mostly deterministic, rather rigid constructions

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and do not assume the produced $\mathbb{Z}^d$ SFT to satisfy any form of mixing. Results for a different kind of subdynamics introduced by Hochman in [7] and strengthened in a recent preprint by Aubrun and Sablik [1] (see also [5] for a different approach to the same problem) show a similar tendency for $\mathbb{Z}^d$ sofic shifts where basically all possible $\mathbb{Z}^{d-1}$ subdynamics are realized. Unfortunately the construction methods developed by those authors are again very rigid. In particular it is not known whether those results are compatible with any kind of (uniform) mixing.

The present paper thus concentrates on the question what happens if we do care about mixing and what kind of $\mathbb{Z}$ subshifts still appear as projective subdynamics of $\mathbb{Z}^d$ SFTs with strong mixing properties. As expected, stronger mixing conditions put more and more restrictions on the existing one-dimensional projective subdynamics: In the case of merely topological mixing $\mathbb{Z}^d$ SFTs this effect is still relatively weak and once more many exotic or pathological $\mathbb{Z}$ subshifts are realizable (see Section 5). The situation however changes drastically in $\mathbb{Z}^d$ SFTs achieving any uniform mixing condition. In fact this influence of uniform mixing is unexpectedly strong and highly restrictive excluding most of the projective subdynamics seen in general $\mathbb{Z}^d$ SFTs and leading to a rather tame, well-understandable framework.

In fact it turns out that for $\mathbb{Z}^d$ SFTs there is a quite weak compatibility condition which we introduce in Section 4. This condition already forces the projective subdynamics seen along a corresponding one-dimensional sublattice to be a $\mathbb{Z}$ sofic system. Depending on the geometry of the rules defining our $\mathbb{Z}^d$ SFT this condition can be met even in the absence of mixing but it always is implied for certain (or all) sublattices by a strong enough uniform mixing property.

A minor observation shows that the presence of uniform mixing (e.g. block gluing) already forces uniform mixing of all lower-dimensional projective subdynamics. This in consequence eliminates all non-trivial zero-entropy systems from being realizable as projective subdynamics and in particular gets rid of the zero-entropy $\mathbb{Z}$ sofics that played such an important role in the classification obtained in [13].

Having established these restrictive results we are basically left with the possible one-dimensional projective subdynamics being mixing $\mathbb{Z}$ sofic systems. The classification then is completed in Section 6 by an explicit construction allowing us to realize arbitrary mixing $\mathbb{Z}$ sofics even inside strongly irreducible $\mathbb{Z}^2$ SFTs. Hence the class of one-dimensional projective subdynamics seen along cardinal directions in block gluing $\mathbb{Z}^2$ (resp. uniformly filling $\mathbb{Z}^d$) SFTs and the class of one-dimensional projective subdynamics realizable along arbitrary directions in (4-corner-gluing $\mathbb{Z}^2$ resp.) strongly irreducible $\mathbb{Z}^d$ SFTs both coincide with the class of mixing $\mathbb{Z}$ sofics. We remark that the alleged conceptual difference of possible projective subdynamics with respect to distinct sublattices – which shows up in this paper for the first time – is just an artefact of the definition of the uniform mixing notions. This is due to the fact that some of those (like block gluing or the uniform filling property) favor cardinal directions instead of treating all directions equally.

In Section 5 we construct some $\mathbb{Z}^2$ SFT examples showing optimality of most of our restrictive results for the various (uniform) mixing conditions. In addition the two most (dimensionally-)degenerate classes of $\mathbb{Z}^d$ SFTs being either the full or a constant $\mathbb{Z}^{d-1}$-extension of some one-dimensional subshift are studied in Section 4. For those two extreme cases we prove that the class of possible one-dimensional projective subdynamics is simply the class of $\mathbb{Z}$ SFTs.

2. Basic definitions

Although we assume some familiarity with symbolic dynamics – for additional background refer to [12] or [9] – we recall a few definitions and notations.

Every finite (discrete) alphabet $\mathcal{A}$ gives rise to a *d-dimensional full shift* $\mathcal{A}^{\mathbb{Z}^d}$ where $d \in \mathbb{N}$. Equipped with the product topology this compact space supports a natural expansive and continuous $\mathbb{Z}^d$ (shift) action $\sigma : \mathbb{Z}^d \times \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ given by translations such that $(\sigma^{\vec{\imath}}(x))_{\vec{\jmath}} = (\sigma(\vec{\imath}, x))_{\vec{\jmath}} := x_{\vec{\imath}+\vec{\jmath}}$ for all $\vec{\imath}, \vec{\jmath} \in \mathbb{Z}^d$, $x \in \mathcal{A}^{\mathbb{Z}^d}$.

A shift-invariant closed subset of $\mathcal{A}^{\mathbb{Z}^d}$ is called a $\mathbb{Z}^d$ *(sub)shift* and a *subsystem* $Y \subseteq X$ of some $\mathbb{Z}^d$ subshift X is again a shift-invariant closed subset of X, together with the restriction $\sigma|_{\mathbb{Z}^d \times Y}$ of the $\mathbb{Z}^d$ shift action to this set. A $\mathbb{Z}^d$ subshift $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ is called *trivial* if it consists of only one point.

Let $\mathcal{A}^{*,d} := \bigcup_{F \subsetneq \mathbb{Z}^d \text{ finite}} \mathcal{A}^F$ denote the countable set of all patterns built from $\mathcal{A}$ on finite subsets of $\mathbb{Z}^d$ where patterns that coincide up to a translation are identified. Every $\mathbb{Z}^d$ subshift on $\mathcal{A}$ is given by specifying a set of *forbidden patterns* $\mathcal{F} \subseteq \mathcal{A}^{*,d}$ such that none of its points contain an element from $\mathcal{F}$ as a subpattern. We use $\mathsf{X}(\mathcal{F}) := \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} \;\middle|\; \forall\, F \subsetneq \mathbb{Z}^d \text{ finite} : \; x|_F \notin \mathcal{F} \right\}$ to denote the corresponding subshift. If $\mathcal{F}$ can be chosen finite, $\mathsf{X}(\mathcal{F})$ is called a *(d-dimensional) shift of finite type* ($\mathbb{Z}^d$ SFT) and we may assume $\mathcal{F} \subseteq \mathcal{A}^F$ for a single non-empty shape $F \subsetneq \mathbb{Z}^d$.

Given $X = \mathsf{X}(\mathcal{F}) \subseteq \mathcal{A}^{\mathbb{Z}^d}$, a (finite) pattern $P \in \mathcal{A}^{*,d}$ is called *locally admissible* in X if it contains no element from $\mathcal{F}$ as a subpattern[1], whereas P is called *globally admissible* in X if it shows up in a valid point of X, i.e. if P can be extended to a configuration on all of $\mathbb{Z}^d$ which does not contain any element from $\mathcal{F}$. (The same terminology is used for (infinite) configurations $C \in \mathcal{A}^I$ where $I \subseteq \mathbb{Z}^d$ is infinite.) The set of all globally admissible (finite) patterns is known as the *language* $\mathcal{L}(X) := \left\{ x|_F \;\middle|\; x \in X \wedge F \subsetneq \mathbb{Z}^d \text{ finite} \right\}$ of the $\mathbb{Z}^d$ shift X. Its subset of patterns with a certain shape $F \subsetneq \mathbb{Z}^d$ will be denoted by $\mathcal{L}_F(X) := \{x|_F \mid x \in X\} \subsetneq \mathcal{L}(X)$.

The most fundamental invariant associated to the language of a $\mathbb{Z}^d$ subshift X is its *(topological) entropy*. This non-negative real number measures the exponential growth rate of the number of globally admissible patterns and is defined as

$$h_{\text{top}}(X) := \lim_{n\to\infty} \frac{\log |\mathcal{L}_{C_n}(X)|}{|C_n|}$$

where $C_n := \{\vec{\imath} \in \mathbb{Z}^d \mid \|\vec{\imath}\|_\infty \le n\}$. (As the inradii of the shapes C_n grow unbounded the above limit exists by generalized subadditivity [2].)

Another highly important concept in the theory of $\mathbb{Z}^d$ subshifts is the notion of *mixing*. Before we can state the corresponding definitions we need some notation:

Let δ_∞ denote the maximum-metric on $\mathbb{Z}^d$, i.e. for every pair $\vec{u}, \vec{w} \in \mathbb{Z}^d$ we define $\delta_\infty(\vec{u}, \vec{w}) := \|\vec{u} - \vec{w}\|_\infty = \max_{1 \le k \le d} |\vec{u}_k - \vec{w}_k|$. There is a natural way to extend δ_∞ to a non-negative symmetric function $\delta_\infty(U, W) := \min_{\vec{u} \in U, \vec{w} \in W} \delta_\infty(\vec{u}, \vec{w})$ yielding the separation between non-empty subsets $U, W \subseteq \mathbb{Z}^d$.

For coordinates $\vec{u}, \vec{w} \in \mathbb{R}^d$ we will denote by $\vec{u} \preceq \vec{w}$ the fact that for every $1 \le k \le d$ their components satisfy $\vec{u}_k \le \vec{w}_k$. For $\vec{u}, \vec{w} \in \mathbb{Z}^d$ with $\vec{u} \preceq \vec{w}$ the notation $B = [\vec{u}, \vec{w}] := \{\vec{v} \in \mathbb{Z}^d \mid \vec{u} \preceq \vec{v} \preceq \vec{w}\}$ is used to refer to a non-empty (finite) *(rectangular/cuboid) solid block* in $\mathbb{Z}^d$ and we let $\vec{\mathbb{1}} := (1, 1, \ldots, 1) \in \mathbb{Z}^d$ denote the element in $\mathbb{Z}^d$ all of whose components equal 1.

[1]The set of locally admissible patterns depends on the chosen set $\mathcal{F}$, so whenever we talk about those patterns we think of $X = \mathsf{X}(\mathcal{F})$ as defined by a certain set $\mathcal{F}$.

Now the definition of mixing in $\mathbb{Z}^d$ shifts (originally introduced as a direct analogue of the corresponding definition for $\mathbb{Z}$ shifts) can be stated as follows:

Definition 2.1. A $\mathbb{Z}^d$ subshift $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ is *(topologically) mixing* if for any pair of non-empty (disjoint) finite subsets $U, W \subsetneq \mathbb{Z}^d$ there exists a separation constant $M_{U,W} \in \mathbb{N}_0$ so that for any $\vec{v} \in \mathbb{Z}^d$ such that the separation $\delta_\infty(U, \vec{v}+W\,) > M_{U,W}$ and any pair of valid points $y, z \in X$ there exists a valid point $x \in X$ such that $x|_U = y|_U$ and $x|_{\vec{v}+W} = z|_{\vec{v}+W}$.

Note that in the case of a $\mathbb{Z}$ shift X this definition reduces to the usual one where given a pair of (finite) words $u, w \in \mathcal{L}(X)$ appearing in X there is a mixing distance $M = M_{u,w} \in \mathbb{N}_0$ such that for all values $m > M$ the existence of a point $x \in X$ with $x|_{[-|u|,0)} = u$ and $x|_{[m,m+|w|)} = w$ is guaranteed.

While in the one-dimensional setting this is the predominantly used mixing property, for $d > 1$ it turns out too weak. Instead the concept of $\mathbb{Z}^d$ mixing splinters into a large number of non-equivalent stronger conditions[2] according to a possible distinction between cardinal and non-cardinal directions and different geometrical shapes used instead of simply one-dimensional words. Here we recall the most common notions that will be used in the remainder of this paper (see the Appendices of [4] for more details about those properties and the relations between them):

Definition 2.2. A $\mathbb{Z}^d$ subshift X

(1) is called *block gluing with gap* $g \in \mathbb{N}_0$ if for any pair of two solid blocks $B_1 = [\vec{u}^{(1)}, \vec{w}^{(1)}], B_2 = [\vec{u}^{(2)}, \vec{w}^{(2)}] \subsetneq \mathbb{Z}^d$ with separation $\delta_\infty(B_1, B_2) > g$ and any pair of valid points $y, z \in X$ there exists a valid point $x \in X$ such that $x|_{B_1} = y|_{B_1}$ and $x|_{B_2} = z|_{B_2}$.
(2) is called *corner gluing (in the NE-corner) with gap* $g \in \mathbb{N}_0$ if for any solid block $B = [\vec{v} + g\vec{\mathbb{1}}, \vec{w}]$ and any corner shape $C^{\mathrm{NE}} = [\vec{u}, \vec{w}] \setminus [\vec{v}, \vec{w}]$ with $\vec{u}, \vec{v}, \vec{w} \in \mathbb{Z}^d$ with $\vec{u}+\vec{\mathbb{1}} \preceq \vec{v} \preceq \vec{w}-g\vec{\mathbb{1}}$ (and thus separation $\delta_\infty(B, C^{\mathrm{NE}}) > g$) and any pair of valid points $y, z \in X$ there exists a valid point $x \in X$ such that $x|_B = y|_B$ and $x|_{C^{\mathrm{NE}}} = z|_{C^{\mathrm{NE}}}$. (Similarly for the remaining corners.)
(3) has the *uniform filling property (UFP) with filling length* $l \in \mathbb{N}_0$ if for any solid block $B = [\vec{u}, \vec{w}] \subsetneq \mathbb{Z}^d$ and any pair of valid points $y, z \in X$ there exists a valid point $x \in X$ with $x|_B = y|_B$ and $x|_{\mathbb{Z}^d \setminus [\vec{u}-l\vec{\mathbb{1}}, \vec{w}+l\vec{\mathbb{1}}]} = z|_{\mathbb{Z}^d \setminus [\vec{u}-l\vec{\mathbb{1}}, \vec{w}+l\vec{\mathbb{1}}]}$.
(4) is called *strongly irreducible with gap* $g \in \mathbb{N}_0$ if for any pair of non-empty (disjoint) finite subsets $U, W \subsetneq \mathbb{Z}^d$ with separation $\delta_\infty(U, W) > g$ and any pair of valid points $y, z \in X$ there exists a valid point $x \in X$ such that $x|_U = y|_U$ and $x|_W = z|_W$.

A $\mathbb{Z}^d$ subshift X is called *block gluing* (resp. *corner gluing*, etc.) if it is block gluing with gap g (resp. corner gluing with gap g, etc.) for some $g \in \mathbb{N}_0$.

Remarks 2.3. In contrast to Definition 2.1 where the separation constant $M_{U,W}$ may depend on the size and shape of U, W all the mixing properties from Definition 2.2 are *uniform* in the sense that the gap/filling length has to be an independently chosen global constant.

In general the uniform mixing conditions of Definition 2.2 are much stronger than mere topological mixing. E.g. in a non-trivial $\mathbb{Z}^d$ shift all of them imply positive entropy whereas topological mixing does not (see [4], Appendices A,B).

[2] All those distinct properties showing up in the literature have proved useful depending on the topological or measure-theoretic question under consideration.

By compactness the definitions of the various uniform mixing conditions stated for finite shapes are equivalent to their versions using infinite shapes of the same form, as long as those shapes remain separated by the same gap/filling length.

We mention a result from [4] which puts those uniform mixing conditions in a linear order from strong to weak (ending with the non-uniform topological mixing):

Observation 2.4. For $g \in \mathbb{N}_0$ and a $\mathbb{Z}^d$ shift X the following implications hold: X is strongly irreducible with gap $g \overset{(1)}{\Longrightarrow} X$ has the uniform filling property with filling length $g \overset{(2)}{\Longrightarrow} X$ is corner gluing (in any corner) with gap $g \overset{(3)}{\Longrightarrow} X$ is block gluing with gap $g \overset{(4)}{\Longrightarrow} X$ is topologically mixing.

Implications (2), (3), and (4) cannot be reversed even if we allow an increase in g. (1) is not reversible for general $\mathbb{Z}^d$ subshifts; its reversibility for SFTs is still open.

A surjective continuous map between two $\mathbb{Z}^d$ subshifts commuting with the shift actions is called *(topological) factor map* and the image of a subshift X under such map is referred to as a *factor* of X. The class of $\mathbb{Z}^d$ *sofic shifts* ($\mathbb{Z}^d$ sofics) is the set of factors of $\mathbb{Z}^d$ SFTs. Obviously this set is closed under factor maps, i.e. factors of sofic shifts are again sofic, and it strictly contains the class of $\mathbb{Z}^d$ SFTs.

It is known that every (one-dimensional) $\mathbb{Z}$ sofic shift $S \subseteq \mathcal{A}^{\mathbb{Z}}$ can be represented by a *finite, directed*[3], *labeled graph* $G = (V_G, E_G, \lambda_G)$, so that $S = \mathsf{S}(G)$ for

$$\mathsf{S}(G) := \left\{ (\lambda_G(e_i))_{i \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}} \,\middle|\, \forall\, i \in \mathbb{Z} :\ e_i \in E_G \,\wedge\, \mathsf{t}_G(e_i) = \mathsf{i}_G(e_{i+1}) \right\} .$$

Here V_G denotes the finite set of vertices, E_G the finite set of directed edges and $\lambda_G :\ E_G \to \mathcal{A}$ the label map. Moreover we have two functions $\mathsf{i}_G, \mathsf{t}_G :\ E_G \to V_G$ which give the initial respectively terminal vertex of an edge.

In addition the graph G presenting S can be chosen to be *right-resolving*, i.e. for every $v \in V_G$ the restriction $\lambda_G|_{\{e \in E_G | \mathsf{i}_G(e) = v\}}$ of the label map is injective.

As in the theory of digraphs we define the in-degree of a vertex $v \in V_G$ as the cardinality of the set $\mathsf{t}_G^{-1}(v) = \{e \in E_G \mid \mathsf{t}_G(e) = v\}$ while its out-degree is $\left|\mathsf{i}_G^{-1}(v)\right|$. Since we may successively remove vertices whose in- or out-degree is zero without changing the corresponding $\mathbb{Z}$ sofic shift, in this paper we assume all graphs to be *essential*, i.e. $\forall\, v \in V_G :\ \left|\mathsf{i}_G^{-1}(v)\right| \geq 1 \,\wedge\, \left|\mathsf{t}_G^{-1}(v)\right| \geq 1$.

A tuple $(e_1, e_2, \ldots, e_n) \in E_G^n$ ($n \in \mathbb{N}$) of edges such that $\mathsf{t}_G(e_i) = \mathsf{i}_G(e_{i+1})$ for all $1 \leq i < n$ is referred to as a *(finite) path* and a path $(e_1, e_2, \ldots, e_n) \in E_G^n$ ($n \in \mathbb{N}$) such that $\mathsf{t}_G(e_n) = \mathsf{i}_G(e_1)$ is a *cycle*. Furthermore a *right-infinite ray* in G is a one-sided sequence $(e_0, e_1, e_2, \ldots) \in E_G^{\mathbb{N}_0}$ such that $\mathsf{t}_G(e_i) = \mathsf{i}_G(e_{i+1})$ for all $i \in \mathbb{N}_0$ and similarly $(\ldots, e_{-3}, e_{-2}, e_{-1}) \in E_G^{-\mathbb{N}}$ with $\mathsf{t}_G(e_{i-1}) = \mathsf{i}_G(e_i)$ for all $i \in -\mathbb{N}$ is called a *left-infinite ray* in G. A bi-infinite sequence of edges $(e_n)_{n \in \mathbb{Z}} \in E_G^{\mathbb{Z}}$ which corresponds to a valid walk in G, i.e. $\forall\, i \in \mathbb{Z} :\ \mathsf{t}_G(e_i) = \mathsf{i}_G(e_{i+1})$, will be called a *bi-infinite path* in G. Slightly abusing notation we extend the domain of the maps $\mathsf{i}_G, \mathsf{t}_G$ and the domain as well as the range of λ_G to (bi-in-)finite paths, cycles and rays in the natural way without introducing new notation.

In the proof of one of our main results we will use Krieger's definition of future sets [10]: For any $\mathbb{Z}$ shift $Y \subseteq \mathcal{A}^{\mathbb{Z}}$ we denote by $Y^- := \{y|_{-\mathbb{N}} \mid y \in Y\}$ the set of left-infinite rays (*"pasts"*), i.e. restrictions of the bi-infinite sequences forming Y to their negative coordinates. Similarly $Y^+ := \{y|_{\mathbb{N}_0} \mid y \in Y\}$ will be the set

[3] If not explicitly specified otherwise, all our graphs will be understood to be finite and directed.

of right-infinite rays (*"futures"*) given as projections of points in Y onto the non-negative coordinates. Now every left-infinite ray $y^- \in Y^-$ gives rise to its *future set* $\mathsf{F}_Y(y^-) := \{y^+ \in Y^+ \mid y^-.y^+ \in Y\}$ which contains exactly those right-infinite rays that can follow y^- producing a valid point in Y.

Associated to the family $\{\mathsf{F}_Y(y^-)\}_{y^- \in Y^-}$ of future sets is a right-resolving labeled (possibly infinite) directed graph $\mathsf{K}(Y) = (V_\mathsf{K}, E_\mathsf{K}, \lambda_\mathsf{K})$ with

$$V_\mathsf{K} := \left\{\mathsf{F}_Y(y^-) \;\middle|\; y^- \in Y^-\right\}$$
$$E_\mathsf{K} := \left\{(\mathsf{F}_Y(y^-), a, \mathsf{F}_Y(y^-\,a)) \;\middle|\; \forall\, y^- \in Y^-,\, a \in \mathcal{A} :\; y^-\,a \in Y^-\right\} \subseteq V_\mathsf{K} \times \mathcal{A} \times V_\mathsf{K}$$
$$\lambda_\mathsf{K} :\; E_\mathsf{K} \to \mathcal{A},\; (v_1, a, v_2) \mapsto a$$

which is usually called *Krieger's future cover* and which yields a presentation of Y.

One of the important results relying on this machinery characterizes $\mathbb{Z}$ sofics as being those $\mathbb{Z}$ shifts for which the number of distinct future sets is finite. Therefore Y is sofic if and only if $\{\mathsf{F}_Y(y^-)\}_{y^- \in Y^-}$ is a finite family if and only if the future cover $\mathsf{K}(Y)$ is a finite graph.

In particular this finiteness of its future cover implies that the mixing distance $M_{u,w}$ in a topologically mixing $\mathbb{Z}$ sofic can be chosen independently of the words u, w. Thus a topologically mixing $\mathbb{Z}$ sofic already satisfies a much stronger uniform mixing property (which may be seen as one reason why in the one-dimensional setting no alternative mixing conditions are needed). As a consequence this implies that each non-trivial topologically mixing $\mathbb{Z}$ sofic has strictly positive entropy.

Finally we recall the definition of $\mathbb{Z}$ *coded systems* first introduced in [3]: Let $\mathcal{C} \subseteq \mathcal{A}^{*,1}$ be a possibly infinite family of finite words over the alphabet $\mathcal{A}$. $\mathcal{C}$ is called a *(uniquely decipherable) code* if any concatenation of elements from $\mathcal{C}$ has a unique factorization, i.e. $\forall\, u_i, w_j \in \mathcal{C}$ $(1 \le i \le I,\, 1 \le j \le J)$: $u_1\, u_2\, \ldots\, u_I = w_1\, w_2\, \ldots\, w_J \implies I = J \wedge \forall\, 1 \le i \le I :\; u_i = w_i$. Now a $\mathbb{Z}$ subshift is a *coded system* if it is given as the closure of the set of all bi-infinite sequences obtained as free concatenations of words from a uniquely decipherable code $\mathcal{C}$. We use

$$\mathsf{C}(\mathcal{C}) := \overline{\mathrm{Orb}\{x \in \mathcal{A}^{\mathbb{Z}} \mid \exists\, (w_i \in \mathcal{C})_{i \in \mathbb{Z}} :\; x = \ldots\, w_{-2}\, w_{-1}\, .\, w_0\, w_1\, \ldots\}}$$

to denote the coded system defined by $\mathcal{C}$.

Note that a $\mathbb{Z}$ coded system is topologically mixing whenever the greatest common divisor of the lengths of all codewords is 1 and that every mixing $\mathbb{Z}$ sofic is coded (where the code $\mathcal{C}$ can be chosen to be a regular language).

3. Main results

The general tendency shown in all our results indicates that stronger (uniform) mixing conditions guarantee greater regularity in the projective subdynamics of $\mathbb{Z}^d$ SFTs excluding most pathologies (zero-entropy $\mathbb{Z}$ sofics, highly non-sofic subshifts) found in [13, 15]. As we will see the presence of strong enough mixing often is sufficient to even force the one-dimensional projective subdynamics to be a mixing $\mathbb{Z}$ sofic, which can be thought of as quite close to having local rules.

Before we can state the results obtained in this paper, we have to recall the concept of projective subdynamics of $\mathbb{Z}^d$ subshifts as used in [13]:

For $d \in \mathbb{N}$ and $1 \le k < d$ let $\mathcal{I} = \{\vec{\imath}^{(1)}, \ldots, \vec{\imath}^{(k)}\}$, $\mathcal{J} = \{\vec{\jmath}^{(1)}, \ldots, \vec{\jmath}^{(d-k)}\} \subsetneq \mathbb{Z}^d$ be two disjoint sets of integer vectors such that $\mathcal{I} \,\dot\cup\, \mathcal{J}$ is a linearly independent set which spans $\mathbb{Z}^d$. Then $L := \mathrm{span}_{\mathbb{Z}}(\mathcal{I}) = \left\langle \vec{\imath}^{(1)}, \ldots, \vec{\imath}^{(k)} \right\rangle_{\mathbb{Z}} \lneq \mathbb{Z}^d$ is called a

k-dimensional sublattice of $\mathbb{Z}^d$. The set $L' := \mathrm{span}_{\mathbb{Z}}(\mathcal{J}) = \langle \vec{\jmath}^{(1)}, \ldots, \vec{\jmath}^{(d-k)} \rangle_{\mathbb{Z}}$ constitutes a complementary $(d-k)$-dimensional sublattice.

Definition 3.1. For any $\mathbb{Z}^d$ subshift $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ and any k-dimensional sublattice $L \lneq \mathbb{Z}^d$ $(1 \leq k < d)$, the *L-projective subdynamics* of X, denoted by

$$\mathsf{P}_L(X) := \{x|_L \mid x \in X\} \subseteq \mathcal{A}^L \ ,$$

is the set of points obtained by restricting elements in X to the lattice L. The pair $(\mathsf{P}_L(X), \sigma|_{L \times \mathsf{P}_L(X)})$ is a $\mathbb{Z}^k$ subshift, i.e. $\mathsf{P}_L(X)$ is a compact subset of $\mathcal{A}^L$ which admits an expansive $\mathbb{Z}^k$ action by restricting the shift action to the family $\{\sigma^{\vec{\imath}}\}_{\vec{\imath} \in L}$.

Remarks 3.2. In what follows we will be focusing mainly on one-dimensional sublattices $L = \langle \vec{v} \rangle_{\mathbb{Z}} \lneq \mathbb{Z}^d$ and we want to stress that in this case the integer vector $\vec{v} \in \mathbb{Z}^d$ generating L has to satisfy the condition: $\gcd\{\vec{v}_k \mid 1 \leq k \leq d\} = 1$.

If $L = \langle \vec{e}_1 \rangle_{\mathbb{Z}} \lneq \mathbb{Z}^d$ is the one-dimensional sublattice generated by the first cardinal vector (i.e. the horizontal axis) we speak of the $\mathbb{Z}$-projective subdynamics of X and write $\mathsf{P}_{\mathbb{Z}}(X)$ instead of $\mathsf{P}_L(X)$.

As $X = \mathsf{X}(\mathcal{F})$ comes with a decreasing sequence of $\mathbb{Z}^k$ subshifts $(X_{L,n} \subseteq \mathcal{A}^L)_{n \in \mathbb{N}_0}$ defined as:

$$X_{L,n} := \left\{ x|_L \;\middle|\; x \in \mathcal{A}^{L^{[C_n]}} \ \wedge \ \forall F \subsetneq L^{[C_n]} \text{ finite} : \ x|_F \notin \mathcal{F} \right\}$$

where $L^{[C_n]} := L + [-n\vec{\mathbb{1}}, n\vec{\mathbb{1}}] = \{\vec{\imath} \in \mathbb{Z}^d \mid \min_{\vec{\jmath} \in L} \|\vec{\imath} - \vec{\jmath}\|_\infty \leq n\}$ is the lattice L extended by n steps along all directions[4], we can furthermore distinguish between stable and unstable projective subdynamics (introduced in [13]) as follows:

Definition 3.3. The L-projective subdynamics $\mathsf{P}_L(X)$ is called *unstable* if the sequence $(X_{L,n})_{n \in \mathbb{N}_0}$ decreases infinitely, i.e. $\forall n \in \mathbb{N}_0 \ \exists n' > n : \ X_{L,n'} \subsetneq X_{L,n}$.

Conversely $\mathsf{P}_L(X)$ is called *stable* if the sequence $(X_{L,n})_{n \in \mathbb{N}_0}$ eventually stabilizes, i.e. $\exists N \in \mathbb{N}_0 \ \forall n \geq N$: $X_{L,n} = X_{L,N}$. In this case $\mathsf{P}_L(X) = X_{L,N}$ and a configuration on L is already globally admissible in X if it can be extended to a locally admissible configuration on $L^{[C_N]}$.

Remarks 3.4. As one of the basic observations in [13] we obtained the fact that whenever the L-projective subdynamics $\mathsf{P}_L(X)$ of some $\mathbb{Z}^d$ SFT X is stable, it has to be sofic. The converse however is not true, as a large class of $\mathbb{Z}$ sofics (characterized in [13]) admits both a stable and an unstable realization as $\mathbb{Z}$-projective subdynamics inside $\mathbb{Z}^2$ SFTs.

Note that whenever a $\mathbb{Z}^k$ subshift can be realized as the stable resp. unstable projective subdynamics inside a $\mathbb{Z}^d$ SFT, then it can be realized as well as the stable resp. unstable projective subdynamics of a $\mathbb{Z}^{d'}$ SFT for any $d' > d$. To see this just consider the full $\mathbb{Z}^{d'-d}$-extension (see Definition 3.5) of the corresponding $\mathbb{Z}^d$ SFT.

Definition 3.5. Let $d < d' \in \mathbb{N}$. A $\mathbb{Z}^{d'}$ subshift X is called a *full $\mathbb{Z}^{d'-d}$-extension* of a $\mathbb{Z}^d$ subshift Y if there exists a d-dimensional sublattice $L \lneq \mathbb{Z}^{d'}$ with a complementary $(d'-d)$-dimensional sublattice $L' \lneq \mathbb{Z}^{d'}$ such that $Y = \mathsf{P}_L(X)$ and

$$X = \prod_{L'} \mathsf{P}_L(X) := \left\{ (y^{(\vec{\jmath})})_{\vec{\jmath} \in L'} \;\middle|\; \forall \vec{\jmath} \in L' : \ y^{(\vec{\jmath})} \in Y \right\} \ .$$

[4] $X_{L,n}$ contains all configurations on the lattice L that are locally valid in the sense that they can be extended to $L^{[C_n]}$ without producing a forbidden pattern. It is obvious from its definition that $X_{L,n+1} \subseteq X_{L,n}$ and that $\mathsf{P}_L(X) = \bigcap_{n=0}^{\infty} X_{L,n}$.

Now we are ready to state our main results classifying the one-dimensional projective subdynamics of uniformly mixing $\mathbb{Z}^d$ SFTs which are summarized in the following theorem (and visualized in Table 1). Details, formal definitions, examples and proofs of each single item are given in the subsequent sections of this paper.

Theorem 3.6. (1) *There are $\mathbb{Z}^2$ (and $\mathbb{Z}^d$) SFTs which are topologically mixing even with linearly growing separation constant but which still show highly non-sofic one-dimensional projective subdynamics along their cardinal directions (Example 5.1 and Remark 5.2).*

(2) *If a $\mathbb{Z}^d$ SFT ($d \geq 2$) is tunneled projectively full with respect to a one-dimensional sublattice $L \lneq \mathbb{Z}^d$ its L-projective subdynamics is a $\mathbb{Z}$ sofic (Theorem 4.3). However this L-projective subdynamics need neither be mixing nor stable (Remark 6.2 and Example 5.3).*

(3) *In a block gluing $\mathbb{Z}^2$ SFT the one-dimensional projective subdynamics along each cardinal direction is a mixing $\mathbb{Z}$ sofic (Proposition 4.6 and Lemma 6.1). However again this sofic projective subdynamics does not have to be stable (Example 5.5). Moreover block gluing $\mathbb{Z}^2$ SFTs can still have non-sofic one-dimensional projective subdynamics along non-cardinal directions (Example 5.7).*

(4) *In a 4-corner gluing $\mathbb{Z}^2$ SFT the one-dimensional projective subdynamics along arbitrary directions are mixing $\mathbb{Z}$ sofics (Proposition 4.7 and Lemma 6.1).*

(5) *For $d > 2$, the one-dimensional projective subdynamics along each cardinal direction in a $\mathbb{Z}^d$ SFT with the uniform filling property is a mixing $\mathbb{Z}$ sofic. The same holds along arbitrary directions under the assumption of strong irreducibility of the $\mathbb{Z}^d$ SFT (Proposition 4.8 and Lemma 6.1).*

(6) *Any mixing $\mathbb{Z}$ sofic can be realized as the stable $\mathbb{Z}$-projective subdynamics in a strongly irreducible $\mathbb{Z}^2$ SFT (Theorem 6.4). The same construction can be realized in a strongly irreducible $\mathbb{Z}^d$ SFT for any $d > 2$.*

$\mathbb{Z}^d$ **SFT**	**mixing property**	**cardinal directions**	**arbitrary directions**
$d \geq 2$	topological mixing (incl. linear separation)	non-sofic possible	non-sofic possible
$d = 2$	block gluing	**mixing $\mathbb{Z}$ sofic** (stable or unstable)	non-sofic possible
$d = 2$	4-corner gluing		**mixing $\mathbb{Z}$ sofic**
$d > 2$	uniform filling	**mixing $\mathbb{Z}$ sofic**	?
$d > 2$	strongly irreducible		**mixing $\mathbb{Z}$ sofic**

Table 1. Overview of results (1) and (3) – (5) from Theorem 3.6 on possible one-dimensional projective subdynamics along cardinal resp. arbitrary directions of (uniformly) mixing $\mathbb{Z}^d$ SFTs depending on the dimension $d \geq 2$ and the $\mathbb{Z}^d$ mixing property.

Remark 3.7. In [13] we proved that at least for $\mathbb{Z}^2$ SFTs satisfying the uniform filling property the projective subdynamics along any direction has to be stable. Hence Theorem 3.6.(3) exhibits another conceptual difference between block gluing and uniform filling (see [16] for an earlier distinction concerning projectional entropy).

We mention three open problems originating from but not covered by our main theorem:

Questions 3.8. Is there a block gluing (or even corner gluing) $\mathbb{Z}^d$ SFT ($d \geq 3$) with non-sofic projective subdynamics along one of its cardinal directions?

Similarly, is there a $\mathbb{Z}^d$ SFT satisfying the uniform filling property which has non-sofic projective subdynamics along some non-cardinal direction? (Such an example would settle – in the negative – the tantalizing question whether for $\mathbb{Z}^d$ SFTs uniform filling already implies strong irreducibility.)

What about stableness of the projective subdynamics in the 4-corner gluing $\mathbb{Z}^2$ or the general $\mathbb{Z}^d$ setting for $d \geq 3$ (to which the proof of our Theorem 7.1 from [13] does not extend, but where block gluing again is not enough to force stableness)?

As an additional result we show (Proposition 4.10) that the class of $\mathbb{Z}$ shifts realizable as projective subdynamics inside $\mathbb{Z}^d$ SFTs being either the full or a constant $(d-1)$-dimensional extension of that one-dimensional system coincides with the class of $\mathbb{Z}$ SFTs. Hence the strongest possible restriction is met in the presence of complete $(d-1)$-dimensional unconstrainedness (full extension) as well as in the case of constantness along a $(d-1)$-dimensional sublattice.

4. A condition forcing soficness

It is known that the one-dimensional projective subdynamics of $\mathbb{Z}^2$ (and $\mathbb{Z}^d$) SFTs in general are extremely diverse and exotic. In particular every limit set of a one-dimensional cellular automaton (CA) can be realized[5] constructing a corresponding $\mathbb{Z}^2$ (or $\mathbb{Z}^d$) SFT which emulates the CA's behavior by using completely deterministic local rules along one cardinal direction. Points in such $\mathbb{Z}^2$ SFTs thus represent (infinite) space-time diagrams of the underlying CA, i.e. consecutive rows are filled with iterates under the CA-map of limit set configurations. Since the entire time-evolution of an initial configuration is forced, this construction produces very rigid, non-mixing $\mathbb{Z}^2$ SFTs. On the other hand in [13] we proved that for $\mathbb{Z}^2$ SFTs uniform filling is sufficient to guarantee soficness of its projective subdynamics. Thus it seems that non-determinism of the SFT's local rules as needed to realize a strong enough (uniform) mixing condition forces the projective subdynamics to be well behaved and not too far from being itself controlled by local rules.

In the following we define a new kind of compatibility condition for $\mathbb{Z}^d$ SFTs which is much weaker than a uniform mixing condition. We show that this condition implies soficness of certain one-dimensional projective subdynamics and we investigate which of the standard mixing properties imply this new condition.

We start by introducing two notions from discrete geometry: Let $B = [\vec{u}, \vec{w}] \subsetneq \mathbb{Z}^d$ be a (finite) solid block in $\mathbb{Z}^d$, then $\widetilde{B} := \left\{\vec{v} \in \mathbb{R}^d \,\middle|\, \vec{u} \preceq \vec{v} \preceq \vec{w}\right\} \subsetneq \mathbb{R}^d$ denotes its *filled in* version. Let $H \subseteq \mathbb{R}^d$ be any subset in $\mathbb{R}^d$. The *discretized B-thickened*

[5] As was shown by Hurd [8] those CA limit sets already include highly non-sofic examples coming from strictly context-free, context-sensitive or even non-recursively enumerable languages.

version of H will be the set

$$H^{[B]} := \big(H + \widetilde{B}\big) \cap \mathbb{Z}^d = \big\{\vec{v} \in \mathbb{Z}^d \;\big|\; \exists\, \vec{h} \in H :\; \vec{h} + \vec{u} \preceq \vec{v} \preceq \vec{h} + \vec{w}\big\}\ .$$

If $B = [-n\vec{\mathbb{1}}, n\vec{\mathbb{1}}]$ ($n \in \mathbb{N}$) we may think of $H^{[B]}$ as the enveloping lattice structure of H consisting of all its "neighbors" in $\mathbb{Z}^d$ up to a distance n with respect to the $\|.\|_\infty$-metric.

In the following let $H \subsetneq \mathbb{R}^d$ denote a hypersurface. A simple geometric lemma then shows that whenever the discretized B-thickened hypersurface $H^{[B]} \subsetneq \mathbb{Z}^d$ splits $\mathbb{Z}^d$ into two non-empty $\mathbb{Z}^d$-connected regions those have to be separated from each other by more than the size of the solid block $B \subsetneq \mathbb{Z}^d$.

Lemma 4.1 (Separation lemma)**.** *Let $H \subsetneq \mathbb{R}^d$ be a $(d{-}1)$-dimensional hypersurface splitting $\mathbb{R}^d$ into two non-empty disjoint connected regions $R^-, R^+ \subsetneq \mathbb{R}^d$ such that $\mathbb{R}^d = R^- \,\dot\cup\, H \,\dot\cup\, R^+$ and let $B := [\vec{0}, \vec{w}] \subsetneq \mathbb{Z}^d$ be a finite solid block.*

For any pair of integer vectors $\vec{\imath}, \vec{\jmath} \in \mathbb{Z}^d \setminus H^{[B]}$ with $\vec{\imath} \in R^-$ and $\vec{\jmath} \in R^+$ there does not exist any translate of B by an integer vector $\vec{k} \in \mathbb{Z}^d$ containing both $\vec{\imath}$ and $\vec{\jmath}$, i.e. $\forall\, \vec{k} \in \mathbb{Z}^d :\; \big|\{\vec{\imath}, \vec{\jmath}\} \cap \big(\vec{k} + B\big)\big| \leq 1$.

Proof. Since $\vec{\imath} \in R^-$ and $\vec{\jmath} \in R^+$ are contained in two disjoint sets we get $\vec{\imath} \neq \vec{\jmath}$ for free and as H is a hypersurface separating R^- from R^+ any path γ connecting $\vec{\imath}$ to $\vec{\jmath}$ in $\mathbb{R}^d$ has to cross H, thus contains at least one intersection point with H.

Define $\vec{l} := (\min\{\vec{\imath}_k, \vec{\jmath}_k\})_{1 \leq k \leq d} \in \mathbb{Z}^d$ to be the coordinatewise minimum of $\vec{\imath}$ and $\vec{\jmath}$ and let $\gamma_{\vec{l},\vec{\imath}} := \big\{(1-\lambda)\vec{l} + \lambda\vec{\imath} \mid 0 \leq \lambda \leq 1\big\}$ be the (straight) line segment from $\vec{l}$ to $\vec{\imath}$ in $\mathbb{R}^d$ and $\gamma_{\vec{l},\vec{\jmath}} := \big\{(1-\lambda)\vec{l} + \lambda\vec{\jmath} \mid 0 \leq \lambda \leq 1\big\}$ the corresponding line segment connecting $\vec{l}$ to $\vec{\jmath}$. As mentioned before every path from $\vec{\imath}$ to $\vec{\jmath}$ intersects H. Hence – as the union of $\gamma_{\vec{l},\vec{\imath}}$ and $\gamma_{\vec{l},\vec{\jmath}}$ is such a path – either $\gamma_{\vec{l},\vec{\imath}}$ or $\gamma_{\vec{l},\vec{\jmath}}$ will contain some point of H. W.l.o.g. assume this intersection occurs on $\gamma_{\vec{l},\vec{\imath}}$ and call the (first) intersection point $\vec{h} \in \gamma_{\vec{l},\vec{\imath}} \cap H \subsetneq \mathbb{R}^d$. (The argument for $\gamma_{\vec{l},\vec{\jmath}}$ is completely symmetric.)

Suppose for a contradiction there is an integer vector $\vec{k} \in \mathbb{Z}^d$ such that $\vec{\imath}, \vec{\jmath} \in \vec{k} + B$ are both contained in the translate of B by $\vec{k}$. This would yield $\vec{k} \preceq \vec{\imath}, \vec{\jmath} \preceq \vec{k} + \vec{w}$. But then $\vec{l}$ is sitting in the solid block $\vec{k} + B$ as well and so does the whole line segment $\gamma_{\vec{l},\vec{\imath}}$ (technically it is a subset of the filled-in version $\vec{k} + \widetilde{B}$ of the translated solid block). Note that by definition of $\vec{l}$ we have $(1-\lambda_1)\vec{l} + \lambda_1\vec{\imath} \preceq (1-\lambda_2)\vec{l} + \lambda_2\vec{\imath}$ for all $0 \leq \lambda_1 \leq \lambda_2 \leq 1$ ($\gamma_{\vec{l},\vec{\imath}}$ is a monotone growing path). In particular using $\lambda_1 = 0, \lambda_{\vec{h}}$ (the parameter corresponding to $\vec{h} \in \gamma_{\vec{l},\vec{\imath}}$) and $\lambda_2 = 1$ this would give $\vec{k} \preceq \vec{l} \preceq \vec{h} \preceq \vec{\imath} \preceq \vec{k} + \vec{w}$ and as $\vec{k} \preceq \vec{h}$ we finally get $\vec{h} \preceq \vec{\imath} \preceq \vec{h} + \vec{w}$ which lets us conclude that $\vec{\imath} \in H^{[B]}$, a contradiction. Therefore the assumed $\vec{k} \in \mathbb{Z}^d$ can not exist, showing our claim. □

Now we can state our compatibility condition (a less formal explication follows):

Definition 4.2. Let $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be a $\mathbb{Z}^d$ SFT and let $L = \langle \vec{v}\rangle_{\mathbb{Z}} \lneq \mathbb{Z}^d$ be a one-dimensional sublattice generated by some integer vector $\vec{v} \in \mathbb{Z}^d$.

L is called *tunneled projectively full* (in X), if the following two conditions hold:

1. There exists a $(d-1)$-dimensional hypersurface $H \subsetneq \mathbb{R}^d$ whose intersection with the straight line $\mathbb{R}\vec{v} \subsetneq \mathbb{R}^d$ consists of exactly one point; lets assume $\mathbb{R}\vec{v} \cap H = \{r\vec{v}\}$ for some fixed $r \in \mathbb{R}$. Further H splits $\mathbb{R}^d$ into two non-empty disjoint connected regions $R^-, R^+ \subsetneq \mathbb{R}^d$ so that $R^- \,\dot\cup\, H \,\dot\cup\, R^+ = \mathbb{R}^d$

and R^- resp. R^+ contains the left- resp. right-infinite half of the line $\mathbb{R}\vec{v}$, i.e. $\{s\vec{v} \mid s < r\} \subseteq R^-$ and $\{s\vec{v} \mid s > r\} \subseteq R^+$.

(2) There exists a finite solid block $B = [\vec{0}, \vec{w}] \subsetneq \mathbb{Z}^d$ such that $X = \mathsf{X}(\mathcal{F})$ can be defined by a particular set of forbidden patterns $\mathcal{F} \subseteq \mathcal{A}^B$ of shape B, and there exists a finite subset $T \subsetneq H^{[B]}$ and a finite family of special points $\{x^{(1)}, x^{(2)}, \ldots, x^{(N)}\} \subseteq X$ ($N \in \mathbb{N}$) such that each configuration that can be seen on the sublattice L is compatible with at least one of the configurations $x^{(n)}|_{H^{[B]}\setminus T}$ ($n \in \{1, 2, \ldots, N\}$). (In other words for every $y \in \mathsf{P}_L(X)$ the set $\{x \in X \mid \exists\, n \in \{1, 2, \ldots, N\} : x|_{H^{[B]}\setminus T} = x^{(n)}|_{H^{[B]}\setminus T} \wedge x|_L = y\}$ has to be non-empty.)

If both the above conditions are satisfied by a sublattice L, the $\mathbb{Z}^d$ subshift X will also be referred to as being *tunneled projectively full with respect to the sublattice* L. As a shorthand we will use the abbreviations L is TPF in X resp. X is L-TPF.

To understand this relatively complicated definition think of condition (1) as a possibility to divide the lattice $\mathbb{Z}^d$ into two infinite pieces containing opposite half-lattices of L separated by a discrete hypersurface of sufficient thickness which by condition (2) – if filled with one of finitely many particular configurations except in a finite region forming a tunnel through the thickened hypersurface – does not put any constraints on the set of globally admissible configurations seen on L.

We emphasize that Definition 4.2 is far from a mixing property as we only require compatibility of all possible configurations seen on the sublattice L with a subconfiguration of some particular point $x^{(n)}$ ($n \in \{1, 2, \ldots, N\}$). In contrast a mixing condition would ask this compatibility for subconfigurations of all points.

The idea behind our definition is to control the amount of information that can be transmitted from the left half of a configuration (on R^-) to its right half (on R^+). Since X is a $\mathbb{Z}^d$ SFT defined by forbidden patterns of shape B, fixing a configuration on the discretized B-thickened hypersurface $H^{[B]}$ strictly separates the information in those two regions (due to Lemma 4.1). Furthermore this "uncouples" the past of a configuration $y \in \mathsf{P}_L(X)$ on L from its future. This fact together with the second condition in Definition 4.2 then puts an upper bound on the information flow from $L \cap R^-$ to $L \cap R^+$ through the finite tunnel T which finally allows us to deduce that $\mathsf{P}_L(X)$ has to be sofic.

Theorem 4.3. *Let $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be a $\mathbb{Z}^d$ SFT and $L \lneq \mathbb{Z}^d$ a one-dimensional sublattice. If L is tunneled projectively full (in X), then $\mathsf{P}_L(X)$ is sofic.*

Proof. We rely on the notation introduced in Definition 4.2, i.e. $H^{[B]} \subsetneq \mathbb{Z}^d$ will denote the discretized B-thickened version of the separating hypersurface $H \subsetneq \mathbb{R}^d$, $T \subsetneq H^{[B]}$ will be the finite set of coordinates composing the tunnel region and $\{x^{(1)}, x^{(2)}, \ldots, x^{(N)}\} \subseteq X$ the finite set of chosen special points.

Suppose the sublattice $L = \langle \vec{v} \rangle_{\mathbb{Z}} \lneq \mathbb{Z}^d$ is generated by the integer vector $\vec{v} \in \mathbb{Z}^d$ and the unique intersection of $\mathbb{R}\vec{v}$ and H occurs at coordinate $r\vec{v} \in \mathbb{R}^d$ ($r \in \mathbb{R}$). Then $L^- := \{l\vec{v} \mid l \in \mathbb{Z} \wedge l < r\}$ is the (left) half of L contained in R^- while $L^+ := \{l\vec{v} \mid l \in \mathbb{Z} \wedge l \geq r\}$ denotes the (right) half contained in $R^+(\dot\cup H)$.

Define $X^* := \bigcup_{n=1}^N X^{(n)}$ with $X^{(n)} := \{x \in X \mid x|_{H^{[B]}\setminus T} = x^{(n)}|_{H^{[B]}\setminus T}\}$ and note that condition (2) of Definition 4.2 assures the projection $\pi_L^* : X^* \to \mathsf{P}_L(X)$, $x \mapsto x|_L$ is still surjective, i.e. $\mathsf{P}_L(X^*) = \mathsf{P}_L(X)$. Now let $\mathcal{T} := \bigcup_{n=1}^N \mathcal{T}^{(n)}$ with $\mathcal{T}^{(n)} := \{x|_T \mid x \in X^{(n)}\} \subseteq \mathcal{L}_T(X)$ be the finite set of patterns that appear in points of X^* on the (finite) set of coordinates that forms the tunnel region T. To prove $Y := \mathsf{P}_L(X)$ sofic we use the notion of future sets introduced by Krieger [10]. (See Section 2 for a short review; we will stick to the notation introduced there.)

Observe that every left-infinite ray $y^- \in Y^-$ gives rise to a non-empty set $\mathcal{T}(y^-) := \{x|_T \mid x \in X^* \wedge x|_{L^-} = y^-\} \subseteq \mathcal{T}$ of patterns on T which are compatible with y^-. Similarly we define $\mathcal{T}(y^+) := \{x|_T \mid x \in X^* \wedge x|_{L^+} = y^+\} \subseteq \mathcal{T}$ to be the non-empty set of patterns compatible with a right-infinite ray $y^+ \in Y^+$. For every left-infinite ray $y^- \in Y^-$ its future set $\mathsf{F}_Y(y^-) = \{y^+ \in Y^+ \mid y^- . y^+ \in Y\} \subseteq Y^+$ consists of all right-infinite rays $y^+ \in Y^+$ for which the concatenation $y^- . y^+$ forms a valid point in Y. Using the conditions on X and the surjectivity of π_L^* those future sets can also be written in the following form:

$$\mathsf{F}_Y(y^-) = \bigcup_{n=1}^{N} \left\{x|_{L^+} \;\middle|\; x \in X^{(n)} \wedge x|_{L^-} = y^-\right\} .$$

Now observe that for each $y^+ \in \mathsf{F}_Y(y^-)$ the existence of a pattern $P \in \mathcal{T}$ such that $\left\{x \in X^{(n)} \;\middle|\; x|_{L^-} = y^- \wedge x|_{L^+} = y^+ \wedge x|_T = P\right\} \neq \emptyset$ for some $n \in \{1, 2, \ldots, N\}$ already implies that the whole set $\left\{x'|_{L^+} \;\middle|\; x' \in X^{(n)} \wedge x'|_T = P\right\}$ is a subset of $\mathsf{F}_Y(y^-)$.

To prove this take any element $x \in X^{(n)}$ with $x|_L = y^- . y^+$ (i.e. $x|_{L^-} = y^-$ and $x|_{L^+} = y^+$) and $x|_T = P$ and take some point $x' \in X^{(n)}$ with $x'|_T = P$. We claim that x and x' can be fused to create another valid point $x'' \in X^{(n)}$ such that $x''|_{R^- \cap \mathbb{Z}^d} = x|_{R^- \cap \mathbb{Z}^d}$ and $x''|_{(H \,\dot\cup\, R^+) \cap \mathbb{Z}^d} = x'|_{(H \,\dot\cup\, R^+) \cap \mathbb{Z}^d}$. First note that $x|_{H^{[B]}} = x'|_{H^{[B]}}$ (they coincide with $x^{(n)}$ on $H^{[B]} \setminus T$ and both see the pattern P on T). Now separation (Lemma 4.1) kicks in using the SFTness of X to stitch together the two halves: A point $\widetilde{x} \in \mathcal{A}^{\mathbb{Z}^d}$ is valid in $X = \mathsf{X}(\mathcal{F})$ if and only if it does not contain any of the forbidden patterns $\mathcal{F} \subseteq \mathcal{A}^B$. Lemma 4.1 tells us that any translate $\vec{k} + B$ of the solid block B by any integer vector $\vec{k} \in \mathbb{Z}^d$ either completely belongs to $(R^- \cap \mathbb{Z}^d) \cup H^{[B]}$ or completely to $(R^+ \cap \mathbb{Z}^d) \cup H^{[B]}$. Hence every pattern of shape B that is seen in x'' already appears either in x or in x' (both valid points in X), thus is globally admissible. This makes x'' a valid point in X. Therefore $x''|_{L^-} = x|_{L^-} = y^-$ and $x''|_{L^+} = x'|_{L^+}$ proving the future set $\mathsf{F}_Y(y^-)$ contains the whole set of right-infinite rays $x'|_{L^+}$ that are compatible with P.

Hence we may write: $\mathsf{F}_Y(y^-) = \bigcup_{n=1}^{N} \bigcup_{P \in \mathcal{T}(y^-)} \left\{x|_{L^+} \;\middle|\; x \in X^{(n)} \wedge x|_T = P\right\}$. In particular for any $y^-, \widetilde{y}^- \in Y^-$ we have the equivalence: $\mathsf{F}_Y(y^-) = \mathsf{F}_Y(\widetilde{y}^-)$ if and only if $\mathcal{T}(y^-) \cap \mathcal{T}^{(n)} = \mathcal{T}(\widetilde{y}^-) \cap \mathcal{T}^{(n)}$ for all $n \in \{1, 2, \ldots, N\}$. Therefore the number of distinct future sets is bounded by the number of (non-empty) subsets of $\mathcal{T}^{(1)} \times \mathcal{T}^{(2)} \times \ldots \times \mathcal{T}^{(N)}$:

$$\left|\left\{\mathsf{F}_Y(y^-) \;\middle|\; y^- \in Y^-\right\}\right| < \prod_{n=1}^{N} 2^{\left|\mathcal{T}^{(n)}\right|} \leq 2^{N \cdot |\mathcal{L}_T(X)|} < \infty .$$

Finally the equivalence between soficness and the finiteness of the family of future sets lets us conclude that $Y = \mathsf{P}_L(X)$ is sofic. □

Remark 4.4. As we have seen in the proof of Theorem 4.3 there is a coarse bound on the number of future sets of the one-dimensional sofic projective subdynamics $\mathsf{P}_L(X)$ realized along a tunneled projectively full sublattice L inside the $\mathbb{Z}^d$ SFT X. In most examples (block gluing or strongly irreducible non-degenerate $\mathbb{Z}^2$ SFTs) this upper bound is not met since there always seems to exist some additional organizational overhead setting up the $\mathbb{Z}^2$ system to produce exactly the $\mathbb{Z}$ sofic projective subdynamics we are interested in. Hence an open problem would be to investigate how much of this overhead really is necessary. Put the other way around: How close to the bound can one get for non-trivial strictly sofic $\mathbb{Z}$ shifts? Fixing the alphabet, the type of the $\mathbb{Z}^2$ SFT, the tunnel size and the number $N \in \mathbb{N}$ of special points, what is a (the) strictly $\mathbb{Z}$ sofic with the maximal number of future sets that can be realized as a one-dimensional projective subdynamics?

Observation 4.5. Note that a $\mathbb{Z}^2$ SFT may have an unstable sofic L-projective subdynamics without being L-TPF. As an example take the $\mathbb{Z}^2$ SFT over $\mathcal{A} = \{0, 1\}$ which checks whether every row in each of its points is from the even shift by disallowing the pattern $1\,0\,1$ and by having local rules forcing the shortening of every run of 0's of length ≥ 2 seen in a row by two (changing its first and last 0 into a 1) in the row above. Moreover allow the spontaneous creation of runs of 0's immediately above long runs of 1's. This assures the extendability of exactly those rows containing a point from the even shift. (E.g. by putting rows of the form $1^\infty.1^\infty$ below and following the "shrinking runs of 0's" rule above.) The $\mathbb{Z}^2$ SFT obtained this way however is not $\mathbb{Z}$-TPF as rows of the form $1^\infty\, 0^n.0^n\, 1^\infty$ force the central patterns of the next n rows above. (Those contain the words $1\,0^{n-i}.0^{n-i}\,1$ with $i \in \{1, 2, \ldots, n\}$ growing.) Increasing $n \in \mathbb{N}$ then contradicts the existence of a merely finite set of special points being compatible with all those rows outside a finite tunnel region. Hence the compatibility condition introduced in Definition 4.2 while being sufficient is not necessary for having sofic projective subdynamics.

Subsequently we will show how different uniform mixing conditions appearing in the literature imply TPF for certain one-dimensional sublattices $L \lneq \mathbb{Z}^d$ in two- resp. higher-dimensional SFTs. Combining those results with Theorem 4.3 we obtain items (3),(4) and (5) of our main theorem 3.6 as immediate corollaries.

As the standard mixing properties are all tailor-made to respect the geometry of $\mathbb{Z}^d$ putting emphasis on the cardinal directions there is no need to work with a general hypersurface $H \subsetneq \mathbb{R}^d$. Instead we can just use a cardinal hyperplane[6] $H_{\vec{e}_i}$ perpendicular to $\vec{e}_i$ for some $i \in \{1, 2, \ldots, d\}$. The tunnel region T will then be a (finite) cuboid block containing the intersection between L and H. Furthermore in the mixing framework it is sufficient to use only one special point (i.e. only one fixed configuration on $H^{[B]} \setminus T$ which is compatible with any configuration on L), and from now on we will call this special point $x^* \in X$.

Proposition 4.6. *In a block gluing $\mathbb{Z}^2$ SFT the cardinal sublattices $L = \langle \vec{e}_i \rangle_{\mathbb{Z}} \lneq \mathbb{Z}^2$ ($i = 1, 2$) are tunneled projectively full.*

Proof. W.l.o.g. let $L = \langle \vec{e}_1 \rangle_{\mathbb{Z}} \lneq \mathbb{Z}^2$ be the horizontal sublattice. The proof for the vertical sublattice $L = \langle \vec{e}_2 \rangle_{\mathbb{Z}}$ is analogous. Assume the $\mathbb{Z}^2$ SFT $X = \mathsf{X}(\mathcal{F}) \subseteq \mathcal{A}^{\mathbb{Z}^2}$ is given by the family $\mathcal{F} \subseteq \mathcal{A}^B$ of forbidden patterns of shape $B = [\vec{0}, \vec{w}] \subsetneq \mathbb{Z}^2$ and assume X is block gluing with gap $g \in \mathbb{N}_0$. To prove that L is tunneled projectively full pick as the hypersurface H the hyperplane $H_{\vec{e}_1} = \mathbb{R}\vec{e}_2$ perpendicular to $\vec{e}_1$ which intersects the sublattice L just at the origin $\vec{0}$. Now check that all conditions of Definition 4.2 are satisfied: $H_{\vec{e}_1}$ separates $R^- := \{r\vec{e}_1 + s\vec{e}_2 \mid r, s \in \mathbb{R} \wedge r < 0\} \subsetneq \mathbb{R}^2$

[6]A cardinal hyperplane in $\mathbb{R}^d$ (or $\mathbb{Z}^d$) is generated by a $(d-1)$-element subset of $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_d\}$. Our notation $H_{\vec{e}_i}$ uses the missing cardinal base vector as a subscript to identify the direction perpendicular to the hyperplane.

from $R^+ := \{r\vec{e}_1 + s\vec{e}_2 \mid r,s \in \mathbb{R} \wedge r > 0\} \subsetneq \mathbb{R}^2$ and R^-, R^+ respectively contain the negative resp. positive horizontal axis. As a tunnel region choose the solid block $T := [(0,-g),(\vec{w}_1,g)] \subsetneq H^{[B]}_{\vec{e}_1}$, select an arbitrary point $x^* \in X$ and take any point $y \in \mathsf{P}_L(X)$. Now we can use block gluing to first construct a valid point $x' \in X$ which satisfies $x'|_L = y$ and $x'|_{\mathbb{Z}\times[g+1,\infty)} = x^*|_{\mathbb{Z}\times[g+1,\infty)}$: Note that L and $\mathbb{Z}\times[g+1,\infty)$ are infinite solid blocks separated by more than the gap size g. Using compactness of X arbitrary globally admissible configurations on L and $\mathbb{Z}\times[g+1,\infty)$ can thus be joined together to produce another valid point in X as claimed. Then applying block gluing a second time there is also a valid point $x \in X$ satisfying: $x|_{\mathbb{Z}\times\mathbb{N}_0} = x'|_{\mathbb{Z}\times\mathbb{N}_0}$ and $x|_{\mathbb{Z}\times(-\infty,-g)} = x^*|_{\mathbb{Z}\times(-\infty,-g)}$ (again the separation of those solid blocks is larger than g). In particular $x|_L = x'|_L = y$ and $x|_{H^{[B]}_{\vec{e}_1}\setminus T} = x^*|_{H^{[B]}_{\vec{e}_1}\setminus T}$ showing the compatibility of $x^*|_{H^{[B]}_{\vec{e}_1}\setminus T}$ and y. □

As we will see in Section 5 block gluing is not enough to force soficness of the projective subdynamics in $\mathbb{Z}^2$ SFTs along arbitrary sublattices. Hence in a block gluing $\mathbb{Z}^2$ SFT not all directions need to be tunneled projectively full which is a consequence of the distinction between cardinal and non-cardinal directions in the definition of block gluingness[7]. Nevertheless a slightly stronger uniform mixing condition already yields soficness along all directions.

A $\mathbb{Z}^2$ shift is called *4-corner gluing*, if it is corner gluing in all four cardinal corners, i.e. Definition 2.2.(2) holds not just for the NE-, but analogously also for the NW-, SE- and SW-corner.

Proposition 4.7. *In any 4-corner gluing $\mathbb{Z}^2$ SFT every one-dimensional sublattice $L \lneq \mathbb{Z}^2$ is tunneled projectively full.*

Proof. The following argument proves that in a $\mathbb{Z}^2$ SFT which is corner gluing in the NW- and SE-corner each one-dimensional sublattice $L = \langle \vec{v} \rangle_{\mathbb{Z}} \lneq \mathbb{Z}^2$ with $\vec{v} \in \mathbb{Z}^2$ an integer vector such that $\vec{v}_1 \cdot \vec{v}_2 \geq 0$ and $|\vec{v}_1| \geq |\vec{v}_2|$ (i.e. the slope of L lies between 0 and 1) is tunneled projectively full.

Fix such a sublattice $L = \langle \vec{v} \rangle_{\mathbb{Z}}$. Denote by $g \in \mathbb{N}_0$ the minimal integer that serves as a gap size for all four cardinal corners (or at least for the NW- and SE-corners in this part). Again assume that the $\mathbb{Z}^2$ SFT $X = \mathsf{X}(\mathcal{F}) \subseteq \mathcal{A}^{\mathbb{Z}^2}$ is given by the family $\mathcal{F} \subseteq \mathcal{A}^B$ of forbidden patterns of shape $B = [\vec{0},\vec{w}] \subsetneq \mathbb{Z}^2$. To fulfill Definition 4.2 put $H := H_{\vec{e}_1} = \mathbb{R}\vec{e}_2$, $T := [-2g\vec{e}_2, \vec{w}_1\vec{\mathbb{1}} + 2g\vec{e}_2] \subsetneq H^{[B]}_{\vec{e}_1}$, choose $x^* \in X$, $y \in \mathsf{P}_L(X)$ both arbitrary and let $x^{(y)} \in X$ be a valid point realizing y, i.e. $x^{(y)}|_L = y$. (Again $L \cap H = \{\vec{0}\}$ and $R^- = \{r\vec{e}_1 + s\vec{e}_2 \mid r,s \in \mathbb{R} \wedge r < 0\}$, $R^+ = \{r\vec{e}_1 + s\vec{e}_2 \mid r,s \in \mathbb{R} \wedge r > 0\}$ both contain one half of the sublattice L.)

As in the previous proof a point $x \in X$ with $x|_L = y$ and $x|_{H^{[B]}_{\vec{e}_1}\setminus T} = x^*|_{H^{[B]}_{\vec{e}_1}\setminus T}$ is constructed in two steps: Denote by $B^{NW} := \{r\vec{e}_1 + s\vec{e}_2 \mid r \leq \vec{w}_1 \wedge s > \vec{w}_1 + 2g\}$ the infinite solid block extending infinitely far into the NW-quarter-plane and let $C^{NW} := \{r\vec{e}_1 + s\vec{e}_2 \mid r > \vec{w}_1 + g \vee s \leq \vec{w}_1 + g\}$ be the corresponding infinite corner shape. Note that $\delta_\infty(B^{NW}, C^{NW}) > g$ and hence using NW-corner gluing there is a point $x' \in X$ with $x'|_{B^{NW}} = x^*|_{B^{NW}}$ and $x'|_{C^{NW}} = x^{(y)}|_{C^{NW}}$ (see the black parts of Figure 1). Similarly define $B^{SE} := \{r\vec{e}_1 + s\vec{e}_2 \mid r \geq 0 \wedge s < -2g\}$ to be

[7]Modifying this definition – demanding to be able to glue together any two globally admissible patterns on two arbitrarily rotated solid blocks separated by a uniform gap instead of requiring this only for solid blocks aligned along the cardinal directions – would give the desired TPF result.

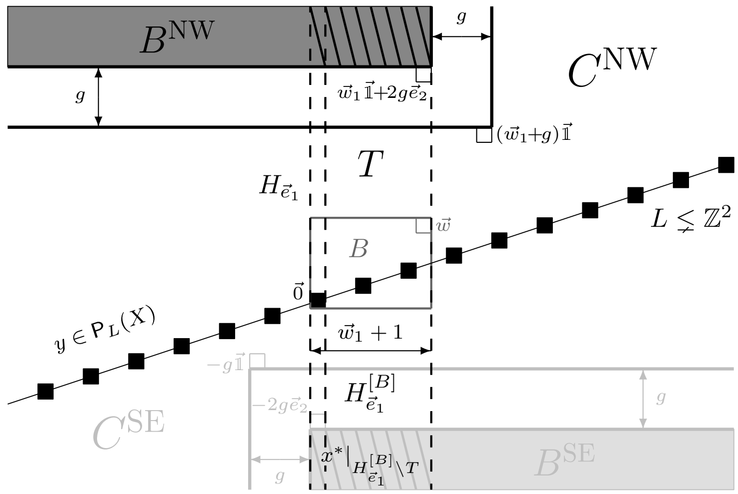


FIGURE 1. Using corner gluing in the NW- and SE-corner to show any given one-dimensional sublattice $L \lneq \mathbb{Z}^2$ with slope between 0 and 1 is tunneled projectively full.

the infinite solid SE-block and $C^{SE} := \{r\vec{e}_1 + s\vec{e}_2 \mid r < -g \ \vee \ s \geq -g\}$ the corresponding infinite corner shape. Again $\delta_\infty(B^{SE}, C^{SE}) > g$ implies the existence of a point $x \in X$ with $x|_{B^{SE}} = x^*|_{B^{SE}}$ and $x|_{C^{SE}} = x'|_{C^{SE}}$ (Figure 1, gray parts). Moreover note that $H_{\vec{e}_1}^{[B]} \setminus T \subsetneq B^{NW} \dot\cup B^{SE}$ and $L \subsetneq C^{NW} \cap C^{SE}$ (as the slope of the sublattice L is between 0 and 1), which yields the claimed properties of x.

For one-dimensional sublattices $L = \langle \vec{v} \rangle_\mathbb{Z} \lneq \mathbb{Z}^2$ with negative slopes between -1 and 0 (i.e. for $\vec{v} \in \mathbb{Z}^2$ such that $\vec{v}_1 \cdot \vec{v}_2 < 0$ and $|\vec{v}_1| \geq |\vec{v}_2|$) we may apply the same argument using corner gluing in the NE- and SW-corners reflecting Figure 1 about its vertical axis. For slopes of modulus larger than 1 a simple reflection of Figure 1 about one of its two principal diagonals shows the corresponding geometrical setup and finishes the proof. □

In the higher-dimensional setting the following results hold:

Proposition 4.8. *For $d > 2$ and every $i \in \{1, 2, \ldots, d\}$, the cardinal sublattice $L = \langle \vec{e}_i \rangle_\mathbb{Z} \lneq \mathbb{Z}^d$ in any $\mathbb{Z}^d$ SFT satisfying the uniform filling property is tunneled projectively full.*

For $d > 2$, every one-dimensional sublattice $L \lneq \mathbb{Z}^d$ in any strongly irreducible $\mathbb{Z}^d$ SFT is tunneled projectively full.

Proof. After the last two propositions the argument for the two higher-dimensional statements should be relatively obvious.

For a $\mathbb{Z}^d$ SFT $X = \mathsf{X}(\mathcal{F})$ given by a family $\mathcal{F} \subseteq \mathcal{A}^B$ of forbidden patterns of shape $B := [\vec{0}, \vec{w}] \subsetneq \mathbb{Z}^d$ satisfying the UFP with filling length $l \in \mathbb{N}_0$ pick any one-dimensional sublattice $L = \langle \vec{e}_i \rangle_\mathbb{Z} \lneq \mathbb{Z}^d$ along a cardinal direction ($i \in \{1, 2, \ldots, d\}$). Let $H := H_{\vec{e}_i}$ be the hyperplane perpendicular to $\vec{e}_i$ and define $T := H_{\vec{e}_i}^{[B]} \cap L^{[C_l]}$ with $C_l := [-l\vec{\mathbb{1}}, l\vec{\mathbb{1}}] = \{\vec{\imath} \in \mathbb{Z}^d \mid \|\vec{\imath}\|_\infty \leq l\}$. The UFP then guarantees – for any choice of $x^* \in X$ and $y \in \mathsf{P}_L(X)$ – the existence of a valid point $x \in X$

with $x|_L = y$ and $x|_{\mathbb{Z}^d \setminus L^{[C_l]}} = x^*|_{\mathbb{Z}^d \setminus L^{[C_l]}}$. As $H^{[B]}_{\vec{e}_i} \setminus T \subsetneq \mathbb{Z}^d \setminus L^{[C_l]}$, in particular $x|_{H^{[B]}_{\vec{e}_i} \setminus T} = x^*|_{H^{[B]}_{\vec{e}_i} \setminus T}$ showing X is L-TPF.

Under the assumption of X being strongly irreducible the same choice of H and T works for any one-dimensional sublattice $L = \langle \vec{v} \rangle_{\mathbb{Z}} \lneq \mathbb{Z}^d$: Just pick $i \in \{1, 2, \dots, d\}$ such that $\vec{v} \in \mathbb{Z}^d$ is not contained in $H_{\vec{e}_i}$ and replace $L^{[C_l]}$ everywhere by its filled in version $\widetilde{L}^{[C_l]} = \mathbb{R}\vec{v} + [-l\vec{\mathbb{1}}, l\vec{\mathbb{1}}]$. As the separation $\delta_\infty(L, H^{[B]}_{\vec{e}_i} \setminus T) > g$ is large enough strong irreducibility yields a way to glue together the globally admissible configuration $x^*|_{H^{[B]}_{\vec{e}_i} \setminus T}$ ($x^* \in X$) with any $y \in \mathsf{P}_L(X)$. □

Remark 4.9. As it is not known whether in the framework of $\mathbb{Z}^d$ SFTs the uniform filling property already implies strong irreducibility – there is no $\mathbb{Z}^d$ SFT example proving the two conditions to be distinct – it may even be the case that for uniform filling $\mathbb{Z}^d$ SFTs there is no need to distinguish between having TPF only along cardinal or along arbitrary one-dimensional sublattices. If there is a difference, again a slight modification of the uniform filling property – using an arbitrarily rotated central solid block and its downsized exterior complement – would yield the more general result about all sublattices being TPF.

To round off this section we prove that whenever a $\mathbb{Z}^d$ SFT is either the full $(d-1)$-dimensional extension (see Definition 3.5) or a constant $(d-1)$-dimensional extension of some $\mathbb{Z}$ subshift seen as its projective subdynamics along a one-dimensional sublattice, then this $\mathbb{Z}$ subshift itself has to be of finite type.

Proposition 4.10. *Let X be a $\mathbb{Z}^d$ SFT and $L \lneq \mathbb{Z}^d$ a one-dimensional sublattice. Denote by $L' \lneq \mathbb{Z}^d$ a $(d-1)$-dimensional sublattice complementary to L and suppose that either*

1. *$X = \prod_{L'} \mathsf{P}_L(X) := \{(y^{(\vec{\jmath})})_{\vec{\jmath} \in L'} \mid \forall \vec{\jmath} \in L' : y^{(\vec{\jmath})} \in \mathsf{P}_L(X)\}$ is the full $(d-1)$-dimensional extension of the $\mathbb{Z}$ subshift $\mathsf{P}_L(X)$ or*
2. *$X = \{(y)_{\vec{\jmath} \in L'} \mid y \in \mathsf{P}_L(X)\}$ is the constant $(d-1)$-dimensional extension of the $\mathbb{Z}$ subshift $\mathsf{P}_L(X)$ with respect to L'.*

Then $\mathsf{P}_L(X)$ is a $\mathbb{Z}$ SFT (and clearly any $\mathbb{Z}$ SFT shows up as $\mathsf{P}_L(X)$ for some X in both these degenerate settings).

Remark 4.11. Note that due to shift invariance of $\mathsf{P}_L(X)$ the full $(d-1)$-dimensional extension does not depend on the choice of L', whereas the constant $(d-1)$-dimensional extension does (as points in $\{(y)_{\vec{\jmath} \in L'} \mid y \in \mathsf{P}_L(X)\}$ are exactly constant along L').

Proof. Let $H_{L'} \subsetneq \mathbb{R}^d$ be the hyperplane spanned by L' and assume our $\mathbb{Z}^d$ SFT $X = \mathsf{X}(\mathcal{F}) \subseteq \mathcal{A}^{\mathbb{Z}^d}$ is given by a family of forbidden patterns $\mathcal{F} \subseteq \mathcal{A}^B$ of common shape $B := [\vec{0}, \vec{w}] \subsetneq \mathbb{Z}^d$. Define a thickened hyperplane $H^{[B]}_{L'} := (H_{L'} + \widetilde{B}) \cap \mathbb{Z}^d$ and denote by $N := \max_{\vec{\jmath} \in L'} \{|r| \mid \vec{\jmath} + r\vec{v} \in H^{[B]}_{L'}\} \in \mathbb{N}_0$ the maximal "distance" along the direction of $L = \langle \vec{v} \rangle_{\mathbb{Z}}$ between any element of L' and the boundary of $H^{[B]}_{L'}$ measured in multiples of the length of $\vec{v}$. Clearly this maximum exists as $\vec{v}$ is not contained in $H_{L'}$ and the solid block B has a finite diameter measured along the direction of L. Specifically we get $H^{[B]}_{L'} \subseteq \{\vec{\jmath} + r\vec{v} \mid \vec{\jmath} \in L' \wedge r \in \mathbb{Z} \wedge |r| \leq N\} \subsetneq \mathbb{Z}^d$. (Note that by definition of a sublattice and its complementary sublattice the volume of the fundamental parallelotope in $\mathbb{R}^d$ spanned by $\vec{v}$ and the generators of L' is one. Hence it can not contain any elements of $\mathbb{Z}^d$ except the ones being its corners.

This then implies that all elements of $H_{L'}^{[B]}$ are actually given as a sum $\vec{\jmath}+r\vec{v}$ with $\vec{\jmath}\in L'$ and $|r|\le N$.)

Suppose the $\mathbb{Z}$ subshift $Y := \mathsf{P}_L(X)$ would not be of finite type. A well-known characterization of "being finite type" [12, Theorem 2.1.8] then assures existence of a non-synchronizing word $w \in \mathcal{L}_{2N+1}(Y)$ of length $2N+1$ and of two points $y, y' \in Y$ with $y|_{[-N,N]} = y'|_{[-N,N]} = w$ but $y|_{(-\infty,0)} \cdot y'|_{[0,\infty)} \notin Y$. Define a point $x \in \mathcal{A}^{\mathbb{Z}^d}$ as follows:

$$\forall\, \vec{\jmath} \in L' : \qquad x|_{\{\vec{\jmath}-r\vec{v}\,|\,r\in\mathbb{N}\}} = y|_{(-\infty,0)} \qquad \text{and} \qquad x|_{\{\vec{\jmath}+r\vec{v}\,|\,r\in\mathbb{N}_0\}} = y'|_{[0,\infty)}\ .$$

Now $x|_{\{\vec{\jmath}-r\vec{v}|\vec{\jmath}\in L'\,\wedge\, r\in\mathbb{N}\}\cup H_{L'}^{[B]}}$ resp. $x|_{\{\vec{\jmath}+r\vec{v}|\vec{\jmath}\in L'\,\wedge\, r\in\mathbb{N}_0\}\cup H_{L'}^{[B]}}$ each look like one half of the point generated by stacking only copies of y resp. y' (i.e. putting one copy on every translate of L by an element of L'). However – in both cases specified in the proposition – such points are by definition elements of X. Therefore neither of the two halves contains a forbidden pattern from $\mathcal{F}$. Their common overlap on $H_{L'}^{[B]}$ then allows to invoke Lemma 4.1 showing the whole point x will not contain any such forbidden pattern. Hence $x \in X$ but $x|_L = y|_{(-\infty,0)} \cdot y'|_{[0,\infty)} \notin \mathsf{P}_L(X)$ yielding an immediate contradiction. □

Hence in the two extreme settings of a $\mathbb{Z}^d$ SFT being completely unconstrained along L' or being constant along L' it is impossible to use local $\mathbb{Z}^d$ rules to get any non-finite-type behaviour along L. Apparently in those degenerate cases there is no room to do computations or invoke any non-local checking procedure neither stable nor unstable. This result stands in stark contrast to the large variety of $\mathbb{Z}$ subshifts realizable in $\mathbb{Z}^d$ SFTs under the presence of partially constrained or even completely deterministic (but non-constant) local rules along L'. E.g. compare the above situation to the $\mathbb{Z}$ sofic projective subdynamics constructed in [13] as well as the (non-sofic) $\mathbb{Z}$ shifts obtainable as limit sets of $\mathbb{Z}$ cellular automata [8].

As demonstrated above, uniform mixing $\mathbb{Z}^d$ SFTs can be seen as an intermediate case where along a one-dimensional sublattice only sofic non-finite-type behaviour can be found. In some sense the presence of a mixing property still limits the space for checking the global structure of rows using the SFT's local rules and thus restricts the capability of realizing too exotic projective subdynamics. As a converse Section 6 shows that nevertheless any mixing $\mathbb{Z}$ sofic can occur even under the strongest uniform mixing condition (i.e. strong irreducibility), given a large enough gap size. This then completes the classification of possible one-dimensional projective subdynamics of uniformly mixing $\mathbb{Z}^d$ SFTs as indicated in Table 1.

5. (Counter-)Examples

In this section we collect several examples showing that most of our results obtained in Section 4 fail if the corresponding hypotheses are not met.

The first example demonstrates that without a uniform mixing condition there is no hope to always get sofic projective subdynamics (even along the cardinal directions). For example many $\mathbb{Z}$ coded systems whose defining codes are non-regular languages can be realized as $\mathbb{Z}$-projective subdynamics in topologically mixing $\mathbb{Z}^2$ SFTs. This is true even for topological mixing with a separation constant $M_{U,W}$ that grows linearly with the size of the shapes $U, W \subsetneq \mathbb{Z}^2$.

Example 5.1 (A topologically mixing $\mathbb{Z}^2$ SFT with linearly growing separation constant whose $\mathbb{Z}$-projective subdynamics is not sofic)**.** Let $\mathcal{A}_1 := \{0, 1, 2\}$ be the alphabet of a $\mathbb{Z}^2$ SFT $X_1 \subsetneq \mathcal{A}_1^{\mathbb{Z}^2}$ whose rows will contain points of the coded system $\mathsf{C}(\mathcal{C}_1)$ given by the uniquely decipherable code $\mathcal{C}_1 := \{0\,1^n\,2^n \mid n \in \mathbb{N}_0\}$. To exact balancedness of adjacent runs of 1's and 2's using only local $\mathbb{Z}^2$ rules we force an almost deterministic evolution between consecutive rows shrinking runs of 1's by one letter from the left and runs of 2's by one letter from the right. The only vertical non-determinism between consecutive rows allows for a choice between filling the space immediately above a long run of 0's either again with 0's (pattern stays unchanged) or by the creation of new runs of 1's and 2's. For now we define $X_1 := \mathsf{X}(\mathcal{F}_1)$ by the following set of forbidden patterns $\mathcal{F}_1$, but will come back to how those local rules enforce the described behaviour in the proof of Claim 5.1.1:

$$\mathcal{F}_1 := \left\{ 0\,2\ ,\ 1\,0\ ,\ 2\,1\ ,\ \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\ ,\ \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\ ,\ \begin{smallmatrix} \ \ 1 \\ 0\,1 \end{smallmatrix}\ ,\ \begin{smallmatrix} 2\,0 \\ 0\,1 \end{smallmatrix}\ ,\ \begin{smallmatrix} 0\ \ \\ 1\,1 \end{smallmatrix}\ ,\ \begin{smallmatrix} 2\ \ \\ 2\,0 \end{smallmatrix}\ ,\ \begin{smallmatrix} 0\,1 \\ 2\,0 \end{smallmatrix}\ ,\ \begin{smallmatrix} \ \ 0 \\ 2\,2 \end{smallmatrix} \right\} .$$

Claim 5.1.1. The $\mathbb{Z}$-projective subdynamics $\mathsf{P}_{\mathbb{Z}}(X_1)$ seen along the horizontal sublattice inside X_1 is the non-sofic mixing $\mathbb{Z}$ coded system $\mathsf{C}(\mathcal{C}_1)$.

Proof. First note that the forbidden patterns $0\,2$, $1\,0$ and $2\,1$ already imply every row in a valid point of X_1 has to contain bi-infinite sequences from the closure of the set of all free concatenations of elements from the family $\{0\} \,\dot\cup\, \{0\,1^m\,2^n \mid m,n \in \mathbb{N}\}$.

The remaining forbidden patterns in $\mathcal{F}_1$ then force the described almost deterministic evolution from one horizontal row to the next one right above: As the only possible vertical extension of a pattern $0\,1^m$ resp. $2^n\,0$ ($m,n \in \mathbb{N}$) is the pattern

$$\begin{matrix} 0\,0\,1^{m-1} \\ 0\,1\,1^{m-1} \end{matrix} \qquad \text{resp.} \qquad \begin{matrix} 2^{n-1}\,0\,0 \\ 2^{n-1}\,2\,0 \end{matrix},$$

this evolution decreases the length of finite (and one-sided infinite) runs of 1's and 2's by exactly one in each step. Thus each word $0\,1^m\,2^n\,0$ is transformed into a word $0\,0\,1^{m-1}\,2^{n-1}\,0\,0$ and this process only stops when $\min\{m,n\} = 0$. Since the case $m > n$ would eventually generate a pattern $1\,0$ while the case $m < n$ would cause a pattern $0\,2$ – neither of which is allowed in X_1 – the only possibility for a word $0\,1^m\,2^n\,0$ to be globally admissible is to have $m = n$ in every step. Hence $\mathsf{P}_{\mathbb{Z}}(X_1)$ is a subsystem of the coded system $\mathsf{C}(\mathcal{C}_1)$.

On the other hand any bi-infinite sequence $y \in \mathsf{C}(\mathcal{C}_1)$ actually appears in $\mathsf{P}_{\mathbb{Z}}(X_1)$ as we can easily construct a valid point $x^{(y)} \in X_1$ realizing y as follows: All rows in the lower half-plane are filled with the point $0^\infty \in \mathsf{C}(\mathcal{C}_1)$, i.e. $\forall\, i < 0$: $x^{(y)}|_{\mathbb{Z}\times\{i\}} = 0^\infty$. As there are no restrictions on symbols that may sit above long runs of 0's we are free to put y in the zeroth row such that $x^{(y)}|_{\mathbb{Z}\times\{0\}} = y$. Here we use the only existing vertical non-determinism – completely unconstrained (local) vertical transitions from a run of at least four 0's to any admissible word in $\mathsf{C}(\mathcal{C}_1)$ starting and ending in 0. Then, depending on the structure of y, the deterministic vertical evolution described above kicks in and fills part of the upper half-plane with triangular shaped regions of 1's resp. 2's sitting right above all runs of 1's resp. 2's in y. The remaining yet unspecified coordinates (if any) can then be filled with 0's to finally get $x^{(y)}$. (Note that the point $x^{(y)}$ can be thought of as the minimal representative realizing y along its zeroth row, in the sense that there are no additional triangular shaped regions of 1's and 2's except those starting in the zeroth row being actually triggered by y. In particular whenever $x^{(y)}|_{[\vec\imath-\vec e_1,\vec\imath+\vec e_1]} = 0\,0\,0$ for some $\vec\imath \in \mathbb{Z}\times\mathbb{N}_0$ then $x^{(y)}|_{\vec\imath+\vec e_2} = 0$ as well.) Thus we have shown $\mathsf{P}_{\mathbb{Z}}(X_1) = \mathsf{C}(\mathcal{C}_1)$.

To prove $\mathsf{C}(\mathcal{C}_1)$ is not sofic we could just mention that $\mathcal{C}_1$ is a non-regular context-free language, but it is also easy to explicitly check that $\left\{\mathsf{F}_{\mathsf{C}(\mathcal{C}_1)}(0^\infty 1^n)\right\}_{n\in\mathbb{N}_0}$ yields an infinite family of pairwise distinct future sets. (Details are left to the reader.) □

Claim 5.1.2. The $\mathbb{Z}^2$ SFT X_1 constructed above is topologically mixing; moreover for any pair of finite shapes $U, W \subsetneq \mathbb{Z}^2$ it is sufficient to take as a separation constant $M_{U,W} = 2 \cdot (\operatorname{diam} U + \operatorname{diam} W + 1)$.

Proof. To prove the claim we show how to extend any globally admissible pattern $P \in \mathcal{L}_U(X_1)$ on a finite shape $U \subsetneq \mathbb{Z}^2$ to a globally admissible pattern $P' \in \mathcal{L}_{U'}(X_1)$ on a finite equilateral triangular shape $U' \subsetneq \mathbb{Z}^2$ (containing U) whose size depends linearly on the diameter of U such that the border of P' consists entirely of 0's.

In fact given U, first choose the smallest solid block $B_U \subsetneq \mathbb{Z}^2$ containing U and assume B_U is a $m \times n$ rectangular block ($m, n \in \mathbb{N}$). Then $\max\{m, n\} = \operatorname{diam} U$ where the diameter is taken in the $\|.\|_\infty$-norm. Now the equilateral triangle U' whose tip is pointing in the $\vec{e}_2$-direction (up) can be chosen to have a base length of $2m + 2n + 2$, a height of $m + n + 1$ and contains B_U in its rows 2 up to $n + 1$.

Such U' is sufficient by the following argument: Every word $w \in \mathcal{L}_m(\mathsf{C}(\mathcal{C}_1))$ of length $m \in \mathbb{N}$ can be extended to a *closed word* (i.e. one starting and ending in 0) $w' \in \mathcal{L}_{2m+2}(\mathsf{C}(\mathcal{C}_1))$ of length $2m+2$. E.g. if w starts with $1^{m'}\,2^{n'}\,0$ ($m' \le n' \in \mathbb{N}_0$) put a prefix $0\,1^{n'-m'}$, if w ends in $0\,1^{m''}\,2^{n''}$ ($m'' \ge n'' \in \mathbb{N}_0$) put a suffix $2^{m''-n''}\,0$ and if $w = 1^{\widetilde{m}}\,2^{m-\widetilde{m}}$ ($0 \le \widetilde{m} \le m$) put a prefix $0\,1^{\max\{0,(m-\widetilde{m})-\widetilde{m}\}}$ and a suffix $2^{\max\{0,\widetilde{m}-(m-\widetilde{m})\}}\,0$. In any case fill the remaining $m - (n' - m') - (m'' - n'') \ge 0$ resp. $m - |m - 2\widetilde{m}| \ge 0$ positions in w' with a prefix of 0's. The worst case being $w = 1^m$ or $w = 2^m$ which would give $w' = 0\,1^m\,2^m\,0 \in \mathcal{L}_{2m+2}(\mathsf{C}(\mathcal{C}_1))$.

Suppose we are given any globally admissible pattern $\widetilde{P} \in \mathcal{L}_{B_U}(X_1)$ on the rectangular block B_U which restricts to the pattern P on $U \subseteq B_U$. The top row of $\widetilde{P}$ then constitutes an admissible word of length m and thus can be extended to a closed word of length $2m+2$. Now the row of $\widetilde{P}$ just below the top row again contains an admissible word of length m. Respecting all vertical transition rules of X_1 we can extend this second row to a closed word of length $2m+2+2$ which together with the extension of the top row still forms a globally admissible pattern in X_1. The additional two symbols are needed to ensure the extended word still starts and ends in 0: Applying the mostly deterministic vertical evolution backwards forces growth of a prefix $0\,1^{m'}\,2$ to $0\,1^{m'+1}\,2$ and growth of a suffix $1\,2^{n'}\,0$ to $1\,2^{n'+1}\,0$; in all other cases we are free to fill those additional and any yet undetermined coordinates of this row with 0's. Carrying this process through all the n rows of the pattern $\widetilde{P}$ gives a globally admissible subpattern of the desired P' on a trapezoidal shape of height n whose top edge has length $2m + 2$ and whose base has length $2m + 2n$. Adding another row of length $2m+2n+2$ at the bottom we can assume the base of our pattern P' consists of the word $0^{2m+2n+2}$. Finally to complete the equilateral triangle U' we apply the vertical evolution upwards starting from the extension of the top row of $\widetilde{P}$. We obtain the pattern P' as claimed (as before we put 0's whenever it is possible). Note that the separation between the original shape U and any coordinate of the border of U' is at most $m + n + 1 \le 2 \cdot \operatorname{diam} U + 1$.

The same can be done for any globally admissible pattern $Q \in \mathcal{L}_W(X_1)$ producing $Q' \in \mathcal{L}_{W'}(X_1)$. Now those triangular shaped patterns P', Q' may be put anywhere in $\mathbb{Z}^2$ as long as they do not overlap, i.e. for all $\vec{v} \in \mathbb{Z}^2$ such that

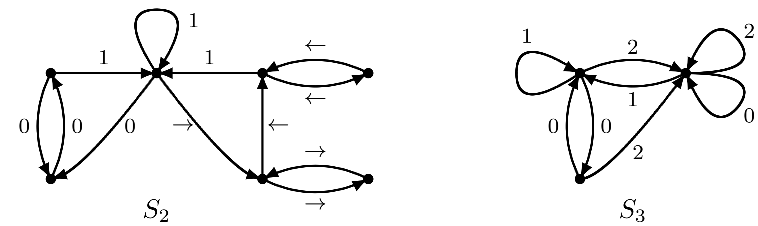


Figure 2. Graph presentations of the mixing $\mathbb{Z}$ sofics S_2 used in Example 5.3 (on the left) and S_3 used in Example 5.5 (on the right).

$\delta_\infty(U', \vec{v}+W') > 0$. Filling all coordinates in $\mathbb{Z}^2 \setminus (U' \,\dot\cup\, \vec{v}+W')$ with 0's will clearly produce a valid point in X_1. Hence $M_{U,W} := 2 \cdot (\operatorname{diam} U + \operatorname{diam} W + 1)$ already guarantees that $\delta_\infty(U, \vec{v}+W) > M_{U,W}$ implies $\delta_\infty(U', \vec{v}+W') > 0$ and thus yields the existence of a valid point $x \in X_1$ with $x|_{U^{(\prime)}} = P^{(\prime)}$ and $x|_{\vec{v}+W^{(\prime)}} = Q^{(\prime)}$. $\square$

Remark 5.2. Similar examples can be constructed using context-sensitive (non-context-free) languages like $\mathcal{C}_1' := \{0\, 1^n\, 2^n\, 3^n \mid n \in \mathbb{N}_0\}$ as a code for the $\mathbb{Z}$ coded system to be realized. Moreover even the (non-synchronized) Dyck shift [11, 6] appears as the $\mathbb{Z}$ projective subdynamics of a topologically mixing $\mathbb{Z}^2$ SFT with linearly growing separation constant (see [15]).

Example 5.3 (A topologically mixing $\mathbb{Z}^2$ SFT with a tunneled projectively full cardinal sublattice whose projective subdynamics along this sublattice is sofic but not stable)**.** Let $\mathcal{A}_2 := \{0, 1, \rightarrow, \leftarrow\}$ be the alphabet of the mixing $\mathbb{Z}$ sofic $S_2 \subsetneq \mathcal{A}_2^{\mathbb{Z}}$ presented by the left labeled graph in Figure 2. This sofic restricts (finite) maximal runs of 0's to have even lengths, while (finite) maximal runs of $\rightarrow$'s and $\leftarrow$'s have to have odd lengths. Moreover a run of $\rightarrow$'s has to be followed by a run of $\leftarrow$'s and this combined run as well as a run of 0's has to be surrounded by 1's. Thus S_2 can also be seen as the (sofic) coded system $\mathsf{C}(\mathcal{C}_2)$ generated by the regular language $\mathcal{C}_2 := \left\{0^{2n}\, 1 \mid n \in \mathbb{N}_0\right\} \,\dot\cup\, \left\{\rightarrow^{2m+1} \leftarrow^{2n+1} 1 \mid m, n \in \mathbb{N}_0\right\}$. Our goal now is to construct a $\mathbb{Z}^2$ SFT $X_2 \subsetneq \mathcal{A}_2^{\mathbb{Z}^2}$ which is $\mathbb{Z}$-TPF and realizes S_2 as its unstable $\mathbb{Z}$-projective subdynamics $\mathsf{P}_{\mathbb{Z}}(X_2)$. We use a still mostly deterministic vertical evolution of rows: Runs of 0's (or 1's) can appear randomly above long runs of 1's but the presence of a word $1\, 0^n\, 1$ in one row exacts a combination of the form $1 \rightarrow^{n'} \leftarrow^{n-n'} 1$ immediately above it where the value of $n' < n \in \mathbb{N}$ can be chosen at random. In the next row, symbols immediately above the transition $\rightarrow\leftarrow$ turn into 1's while the remaining symbols $\rightarrow$ and $\leftarrow$ return to be 0's, thus breaking the original run of 0's in two shorter runs two rows above. (Things then repeat.) Again we define $X_2 := \mathsf{X}(\mathcal{F}_2)$ formally by a set of forbidden patterns $\mathcal{F}_2$ and will show why this gives a $\mathbb{Z}^2$ SFT with the desired properties in Claims 5.3.1 and 5.3.2.

$$\mathcal{F}_2 := \Big\{ 0 \rightarrow,\ \leftarrow 0,\ 0 \leftarrow,\ \rightarrow 0,\ 1 \leftarrow,\ \rightarrow 1,\ \leftarrow\rightarrow,\ \begin{smallmatrix}0\\0\end{smallmatrix},\ \begin{smallmatrix}1\\0\end{smallmatrix},\ \begin{smallmatrix}\leftarrow\\1\end{smallmatrix},\ \begin{smallmatrix}\rightarrow\\1\end{smallmatrix},$$
$$\begin{smallmatrix}\rightarrow\\\rightarrow\end{smallmatrix},\ \begin{smallmatrix}\leftarrow\\\rightarrow\end{smallmatrix},\ \begin{smallmatrix}\rightarrow\\\leftarrow\end{smallmatrix},\ \begin{smallmatrix}\leftarrow\\\leftarrow\end{smallmatrix},\ \begin{smallmatrix}0&\\1&\rightarrow\end{smallmatrix},\ \begin{smallmatrix}&0\\\leftarrow&1\end{smallmatrix},\ \begin{smallmatrix}&0\\\rightarrow&\leftarrow\end{smallmatrix},\ \begin{smallmatrix}0&\\\rightarrow&\leftarrow\end{smallmatrix},\ \begin{smallmatrix}1&\\\rightarrow&\rightarrow\end{smallmatrix},\ \begin{smallmatrix}&1\\\leftarrow&\leftarrow\end{smallmatrix} \Big\}.$$

Claim 5.3.1. The $\mathbb{Z}$-projective subdynamics $\mathsf{P}_{\mathbb{Z}}(X_2)$ seen along the horizontal sublattice in X_2 is the $\mathbb{Z}$ sofic S_2 and this projective subdynamics is unstable.

Proof. The first 7 of the forbidden patterns in $\mathcal{F}_2$ exclude all words of length 2 over $\mathcal{A}_2$ which do not show up in S_2. Hence $\mathsf{P}_{\mathbb{Z}}(X_2)$ is already contained in the 1-step SFT-approximation of this $\mathbb{Z}$ sofic. To obtain $\mathsf{P}_{\mathbb{Z}}(X_2) \subseteq S_2$ it just remains to show that (finite) maximal runs of 0's, $\rightarrow$'s and $\leftarrow$'s have to have the correct parity (i.e. the only additional constraint for S_2). This is done using the other elements of $\mathcal{F}_2$ which compel the quite rigid vertical evolution of locally admissible patterns seen on a horizontal block of coordinates described above: A word $w = 1\,0^n\,1 \in \mathcal{A}_2^{n+2}$ enforces a word $1 \rightarrow^{n'} \leftarrow^{n-n'} 1$ ($n > n' \in \mathbb{N}$) in the row right above it which in turn has as its unique extension to the next row the word $1\,0^{n'-1}\,1\,1\,0^{n-n'-1}\,1$ (and this procedure continues). Suppose the number of consecutive 0's in w is odd, then either n' or $n - n'$ is even while the other one is odd. But then either $n' - 1$ or $n - n' - 1$ is odd again. Since the length of subwords purely composed of 0's ($\rightarrow$'s or $\leftarrow$'s) strictly decreases in every step and since the pattern $1\,0\,1 \in \mathcal{A}_2^3$ not having any valid extension to the next row above is implicitly excluded from $\mathcal{L}(X_2)$, this number theoretic length-modulo-two argument gives an invariant conserved through the whole vertical evolution. In consequence this leads to only allowing the correct parities, i.e. $n \in 2\mathbb{N}$ and $n' \in 2\mathbb{N} - 1$ and shows $\mathsf{P}_{\mathbb{Z}}(X_2) \subseteq S_2$. To prove the converse inclusion given any valid point $s \in S_2$ we define a special point $x^{(s)} \in X_2$ with $x^{(s)}|_{\mathbb{Z}\times\{0\}} = s$ as follows: For any $i \in \mathbb{Z}$ such that $x^{(s)}|_{(i,0)} = s_i \in \{\rightarrow, \leftarrow\}$ put $x^{(s)}|_{(i,-1)} := 0$ while $x^{(s)}|_{(i,-1)} := 1$ in the remaining cases, i.e. if $x^{(s)}|_{(i,0)} = s_i \in \{0, 1\}$. Then for all $i, j \in \mathbb{Z}$ with $j \leq -2$ put $x^{(s)}|_{(i,j)} := 1$ which can be checked to give a locally admissible configuration on the horizontal axis and all the lower half-plane $\mathbb{Z} \times -\mathbb{N}$. On the upper half-plane we then use strictly deterministic rules to complete the (globally) valid point $x^{(s)}$: For every word $1\,0^{2n}\,1$ seen in $x^{(s)}|_{\mathbb{Z}\times\{0\}} = s$ or on the upper half-plane put a word $1 \rightarrow \leftarrow^{2n-1} 1$ above it which then forces a word $1\,1\,1\,0^{2n-2}\,1$ as usual and do the same for rays that contain one-sided infinite runs of 0's (i.e. $1\,0^\infty$ is extended by $1 \rightarrow \leftarrow^\infty$ and $0^\infty\,1$ by $\leftarrow^\infty 1$). Hence after the first step – there might be some occurrences of $1 \rightarrow^m$ with $m > 1$ in s sitting in the zeroth row – the symbol $\rightarrow$ is always preceded by a 1 and followed by a $\leftarrow$. This construction allows to extend $x^{(s)}|_{\mathbb{Z}\times\{0\}} = s$ taking care of all conditions posed by symbols 0, $\rightarrow$ and $\leftarrow$ in s. Finally we may fill yet unconstrained coordinates in $x^{(s)}$ with 1's generating a valid point in X_2. This proves $\mathsf{P}_{\mathbb{Z}}(X_2) = S_2$.

For $n \in \mathbb{N}_0$ apply the same deterministic evolution to the bi-infinite sequence $1^\infty\,.\,0^{2n+1}\,1^\infty \in \mathcal{A}_2^{\mathbb{Z}}$ which is not a valid point in S_2 but still constitutes a locally admissible configuration in X_2. It takes n double-steps to eventually produce the pattern 101 which can not be extended further. Hence such configuration may not be found to be globally inadmissible by the local rules in $\mathcal{F}_2$ unless it is extended by at least $2n$ rows above (and below). This shows that the sequence $\big((X_2)_{\mathbb{Z},2n}\big)_{n\in\mathbb{N}_0}$ of one-dimensional subshifts decreases infinitely often without reaching $\mathsf{P}_{\mathbb{Z}}(X_2)$ for any finite $n \in \mathbb{N}_0$. Therefore the $\mathbb{Z}$-projective subdynamics of X_2 is unstable. □

Claim 5.3.2. The $\mathbb{Z}^2$ SFT X_2 is $\mathbb{Z}$-TPF, topologically mixing but not block gluing.

Proof. For the first statement note that $X_2 = \mathsf{X}(\mathcal{F}_2)$ can as well be defined by a slightly larger set of forbidden patterns $\mathcal{F}_2' \subseteq \mathcal{A}_2^B$ of common shape $B := [\vec{0}, \vec{\mathbb{1}}] \subsetneq \mathbb{Z}^2$. Now take $H := H_{\vec{e}_1} = \mathbb{R}\vec{e}_2$ to be the vertical axis which then gives the thickened hyperplane $H_{\vec{e}_1}^{[B]} = [0,1] \times \mathbb{Z}$. As a tunnel region pick $T := [0,1] \times [-1,4] \subsetneq H_{\vec{e}_1}^{[B]}$ and let $x^* := 1^{\mathbb{Z}^2} \in X_2$ be the fixed point of all 1's.

To show any point $s \in S_2 = \mathsf{P}_{\mathbb{Z}}(X_2)$ is compatible with $x^*|_{H^{[B]}_{\vec{e}_1}\setminus T}$ construct a point $x \in X_2$ with $x|_{\mathbb{Z}\times\{0\}} := s$ fixed and $x|_{H^{[B]}_{\vec{e}_1}\setminus T} = x^*|_{H^{[B]}_{\vec{e}_1}\setminus T}$ wanted. Fill the lower half plane of x as done for $x^{(s)}$ in Claim 5.3.1. In particular $x|_{\mathbb{Z}\times\{j\}} = 1^\infty$ for all $j \leq -2$ which already instantiates the lower half of the configuration $x^*|_{H^{[B]}_{\vec{e}_1}\setminus T}$. For the upper half we have to work a little harder. First to avoid generation of unnecessary occurrences of 0's (not forced by s) whenever $x|_{\vec{\imath}} = 1$ for some $\vec{\imath} \in \mathbb{Z}\times\mathbb{N}_0$ put $x|_{\vec{\imath}+\vec{e}_2} := 1$ as well. Now suppose $x|_{[0,2]\times\{0\}} = 1 \rightarrow\rightarrow$ (resp. $x|_{[-1,1]\times\{0\}} = \leftarrow\leftarrow 1$). Filling the next row in a legal way gives $x|_{[0,1]\times\{1\}} = 1\,0$ (resp. $0\,1$) as our only option. Then put down the next row choosing $x|_{[0,2]\times\{2\}} = 1 \rightarrow\leftarrow$ resp. $x|_{[-1,1]\times\{2\}} = \rightarrow\leftarrow 1$ to finally arrive at $x|_{[0,1]\times\{3\}} = 1\,1$ which then is kept to meet the upper half of the configuration $x^*|_{H^{[B]}_{\vec{e}_1}\setminus T}$ while filling in the remaining rows. Next assume $x|_{[0,1]\times\{0\}} \in \{\rightarrow\rightarrow, \leftarrow\leftarrow\}$ which either leads to $x|_{[0,1]\times\{1\}} \in \{0\,1, 1\,0\}$ (and we are back to the above chain) or to $x|_{[0,1]\times\{1\}} = 0\,0$. At this point there are two possibilities: If neither $x|_{(-\infty,0]\times\{1\}}$ ends in nor $x|_{[1,\infty)\times\{1\}}$ starts with an even number of 0's we may choose the next row to satisfy $x|_{[0,1]\times\{2\}} = \rightarrow\leftarrow$ to get $x|_{[0,1]\times\{3\}} = 1\,1$ and continue as above. In the complementary case there is always an extension to the next row giving $x|_{[-1,1]\times\{2\}} = \rightarrow\leftarrow\leftarrow$ which then yields $x|_{[0,1]\times\{3\}} = 1\,0$ and retracing the last two steps of the first extension we arrive at $x|_{[0,1]\times\{5\}} = 1\,1$ (this being the longest chain of rows needed). Note that all valid subwords of length 2 that can appear on $x|_{[0,1]\times\{0\}}$ are covered in one of the constructions above. Hence starting from an arbitrary row s and carrying out (part of) the corresponding chain we always get $x|_{[0,1]\times\{j\}} = 1\,1$ for all $j \geq 5$ thus realizing the upper half of the desired configuration $x^*|_{H^{[B]}_{\vec{e}_1}\setminus T}$.

For the mixing statement just observe that to go from a legal row $1^\infty\,.\,0^{2^n}\,1^\infty$ to a row $1^\infty\,.\,1^{2^n}\,1^\infty$ even with the fastest divide and conquer strategy – changing 2^m 0's into 1's while extending from row $2(m-1)$ to row $2m$ above ($m \in \mathbb{N}$) – takes at least n double-steps. This immediately contradicts block gluing. Nevertheless we remark that X_2 is topologically mixing (by a similar argument as in Claim 5.1.2; not explicitly shown here). □

Remark 5.4. Example 5.3 in particular proves that there is no converse to Proposition 4.6 since tunneled projective full-ness is far from giving a block gluing SFT (it doesn't even imply topological mixing). The reason for this vaguely shows up at the end of Claim 5.3.2: In the tunneled projectively full $\mathbb{Z}^2$ SFT X_2 not all points may be used as x^* together with a fixed (finite) tunnel region. In fact only specific configurations on $H^{[B]} \setminus T$ are compatible with arbitrary points $s \in S_2 = \mathsf{P}_{\mathbb{Z}}(X_2)$ seen on the horizontal sublattice whereas in a block gluing $\mathbb{Z}^2$ SFT any point $x \in X_2$ would do the trick.

By employing a mixture of a stable and an unstable recognition/checking procedure, the next example shows how to avoid stability of the projective subdynamics (seen on a cardinal sublattice) even in block gluing $\mathbb{Z}^2$ SFTs.

Example 5.5 (A block gluing $\mathbb{Z}^2$ SFT with a sofic unstable projective subdynamics along its horizontal sublattice)**.** Let $\mathcal{A}_3 := \{0, 1, 2\}$ be the alphabet of the mixing $\mathbb{Z}$ sofic $S_3 \subsetneq \mathcal{A}_3^{\mathbb{Z}}$ presented by the right labeled graph in Figure 2. Observe that S_3 differs from the full shift on three symbols only by forbidding all words of the form $1\,0^{2n+1}\,1$ with $n \in \mathbb{N}_0$, i.e. (finite) maximal runs of 0's surrounded by 1's have to have even length. We want to realize S_3 as the unstable $\mathbb{Z}$-projective subdynamics $\mathsf{P}_{\mathbb{Z}}(X_3)$ along the horizontal sublattice in a $\mathbb{Z}^2$ SFT $X_3 \subsetneq \mathcal{A}_3^{\mathbb{Z}^2}$. To get X_3 block gluing we implement a non-deterministic vertical evolution that can choose between slowly decreasing the length of a run of 0's (by 2 in each consecutive row) and an instant parity check (using a subpattern of the periodic point $(1\,0\,0\,1)^\infty$).

Define $X_3 := \mathsf{X}(\mathcal{F}_3) \subsetneq \mathcal{A}_3^{\mathbb{Z}^2}$ by the following set of forbidden patterns:

$$\mathcal{F}_3 := \Big\{ 1\,0\,1\,,\; 1\,0\,0\,0\,1\,,\; \begin{smallmatrix}2\\0\end{smallmatrix}\,,\; \begin{smallmatrix}0\\2\end{smallmatrix}\,,\; \begin{smallmatrix}0\,1\,0\\0\,0\,0\end{smallmatrix}\,,\; \begin{smallmatrix}1\,1\,1\\0\,0\,0\end{smallmatrix}\,,\; \begin{smallmatrix}0\,0\,0\,1\,1\\0\,0\,0\,0\,0\end{smallmatrix}\,,\; \begin{smallmatrix}1\,1\,0\,0\,0\\0\,0\,0\,0\,0\end{smallmatrix}\,,$$
$$\begin{smallmatrix}0\;\;\\1\,0\end{smallmatrix}\,,\; \begin{smallmatrix}\;\;0\\0\,1\end{smallmatrix}\,,\; \begin{smallmatrix}\;\;0\\1\,0\end{smallmatrix}\,,\; \begin{smallmatrix}0\;\;\\0\,1\end{smallmatrix}\,,\; \begin{smallmatrix}\;\;1\;\;\;\;\\1\,0\,0\,0\end{smallmatrix}\,,\; \begin{smallmatrix}\;\;\;\;1\;\;\\0\,0\,0\,1\end{smallmatrix} \Big\}\,.$$

Claim 5.5.1. The $\mathbb{Z}$-projective subdynamics $\mathsf{P}_{\mathbb{Z}}(X_3)$ seen along the horizontal sublattice in X_3 is the $\mathbb{Z}$ sofic S_3 and this projective subdynamics is unstable.

Proof. To prove $\mathsf{P}_{\mathbb{Z}}(X_3) \subseteq S_3$ it suffices to show that none of the words $1\,0^{2n+1}\,1$ ($n \in \mathbb{N}_0$) is globally admissible in X_3 which follows from a simple observation: Note that in X_3 there are basically two ways to extend a pattern $1\,0^n\,1$ ($2 \le n \in \mathbb{N}$) to the next row above. One evolution, which we will call *slow decay*, extends such a pattern with the word $1\,0^{n-2}\,1$, shortening the runs of 0's by one from the left and one from the right thus not checking the actual length of the block of 0's but rather keeping its length modulo 2 invariant and thus causing the projective subdynamics to be unstable. The other evolution, which we will call *instant check*, only applies to patterns with $n \in 4\mathbb{N}$ (a multiple of four) and puts down a $\frac{n}{4}$-fold concatenation of the word $1\,0\,0\,1$ above the block of 0's to immediately check the run's even length making X_3 block gluing. Below we show the two evolution rules (the reader is invited to check that the set $\mathcal{F}_3$ allows exactly those two cases):

$$\textit{slow decay:}\quad \begin{smallmatrix}1\,0\,0\ldots0\,0\,1\\1\,0\,0\,0\ldots0\,0\,0\,1\end{smallmatrix} \quad\text{vs.}\quad \textit{instant check:}\quad \begin{smallmatrix}1\,0\,0\,1\,1\,0\,0\,1\,1\ldots0\,0\,1\,1\,0\,0\,1\\1\,0\,0\,0\,0\,0\,0\,0\,0\,0\ldots0\,0\,0\,0\,0\,0\,0\,1\end{smallmatrix}$$

Now for a word $1\,0^{2n+1}\,1$ ($n \in \mathbb{N}_0$) only slow decay is possible, eventually leading to a forbidden pattern $1\,0\,0\,0\,1$ (or $1\,0\,1$). Therefore none of the forbidden words defining S_3 is globally admissible in X_3 and therefore $\mathsf{P}_{\mathbb{Z}}(X_3) \subseteq S_3$.

For the converse inclusion, as usual, we construct a point $x^{(s)} \in X_3$ realizing an arbitrary $s \in S_3$ along its zeroth row: Put $x^{(s)}|_{\mathbb{Z}\times\{0\}} := s$ and $x^{(s)}|_{\mathbb{Z}\times\{-j\}} := 1^\infty$ for all $j \in \mathbb{N}$. In the upper half-plane let blocks $1\,0\,0\ldots$ and $\ldots0\,0\,1$ evolve by using slow decay and let all symbols 1 and 2 propagate infinitely, i.e. for all $\vec{\imath} \in \mathbb{Z}\times\mathbb{N}_0$ if $x^{(s)}|_{\vec{\imath}} = 1$ put $x^{(s)}|_{\vec{\imath}+\vec{e}_2} := 1$ and if $x^{(s)}|_{\vec{\imath}} = 2$ put $x^{(s)}|_{\vec{\imath}+\vec{e}_2} := 2$ as well. This gives a valid point in X_3. Thus $\mathsf{P}_{\mathbb{Z}}(X_3) = S_3$ and moreover the projective subdynamics is unstable as the slow decay of a configuration $1^\infty\,.\,0^{2n+5}\,1^\infty \in \mathcal{A}_3^{\mathbb{Z}}$ ($n \in \mathbb{N}_0$) takes exactly $(n+1)$-rows (above) to produce a forbidden pattern and extending such a configuration below adding rows $1^\infty \in S_3$ is no problem. Hence $1^\infty\,.\,0^{2n+5}\,1^\infty \in (X_3)_{\mathbb{Z},n} \setminus \mathsf{P}_{\mathbb{Z}}(X_3)$ shows that the sequence $\big((X_3)_{\mathbb{Z},n}\big)_{n\in\mathbb{N}_0}$ does not stabilize. □

Claim 5.5.2. The $\mathbb{Z}^2$ SFT X_3 is block gluing but does not have the UFP.

Proof. Observe that any globally admissible pattern $P \in \mathcal{L}_B(X_3)$ on a solid block $B = [\vec{u},\vec{w}] \subsetneq \mathbb{Z}^2$ can be extended to a still globally admissible pattern $P' \in \mathcal{L}_{B'}(X_3)$ on the solid block $B' := [\vec{u}-\vec{\mathbb{1}}, \vec{w}+\vec{\mathbb{1}}+2\vec{e}_2]$ such that $P'|_B = P$ as indicated in Figure 3. To define P' first attach a column of 2's to the left and to the right of P and a row of 1's beneath the bottom of this extension of P. The remaining three rows on top of P are then filled by immediately using the instant check evolution on all runs of 0's seen in the top row of P where it can be applied (everywhere except on words $1\,0^{4n+2}\,1$ with $n \in \mathbb{N}_0$) and one step of the slow decay rule on those remaining runs $1\,0^{4n+2}\,1$ transforming them into words $1\,0^{4n}\,1$ where the instant check rule can be applied one row further above. All other parts (symbols 1's and 2's) of the top row of P are just copied without any changes. To finish the construction of P' its top row is again completely filled with 1's. It should be obvious that two such extensions P' and Q' (coming from globally admissible rectangular patterns $P, Q \in \mathcal{L}(X_3)$) can be placed next to each other without violating any rules in X_3. Filling all remaining coordinates with 1's then produces a valid point in X_3 and proves block gluingness with gap $g = 3$ (horizontally a separation > 1 would be sufficient, but vertically the separation for some patterns has to be at least > 3).

$P' :=$

```
1  11111111111111111111111111111111111111111111111111111  1
2  11111212211111111111111111111111211110011001112111111  2
2  11111212211001100110011111110011211100000000112100110  2
2 |01001202210000000000001111100001211000000000012000000| 2
2 |11111202100000000000000111111111210000000000001111111| 2
2 |12111202111111111111111100110211100000000000000122202| 2
2 |01100212001100111221111000011111000000000000000012211| 2
2 |00111212000000011222110000001110000000000000000001111| 2
2 |11111112111111122111100000000111111111111111111112210| 2
2 |10011001100110011001111111111211201100111001112021100| 2
2 |00000000000000000000121200100110202200011111221111000| 2
1  11111111111111111111111111111111111111111111111111111  1
```

FIGURE 3. Extending a given rectangular pattern $P \in \mathcal{L}(X_3)$ (inside the rectangular box) to a slightly larger rectangular pattern $P' \in \mathcal{L}(X_3)$ forcing its border to consist entirely of 1's and 2's.

Clearly X_3 does not have the UFP as any rectangular border of width 3 which is completely filled with 0's forces a unique interior consisting entirely of 0's. □

Remark 5.6. Modifying the local rules of Example 5.5 to have our *slow decay* evolution acting only from the left and introducing the option of a *partial instant check* from the right, i.e.

one-sided slow decay from the left:

```
     110...0001
   11000...0001
 100000...0001
```

partial instant check from the right:

```
     110...00000111111111
   11000...00000110011001
 100000...00000000000001   ,
```

we can produce a NE-corner gluing $\mathbb{Z}^2$ SFT X_3' with a still unstable sofic projective subdynamics $\mathsf{P}_{\mathbb{Z}}(X_3') = S_3$ along its horizontal sublattice.

As we have seen in Proposition 4.6 for $\mathbb{Z}^2$ SFTs block gluing is enough to ensure soficness of the projective subdynamics along the cardinal directions. However we could not extend this result to get sofic projective subdynamics along all directions without strengthening the uniform mixing condition in Proposition 4.7. In fact there is a reason for the necessity of the stronger hypothesis as the subsequent final example provides a block gluing $\mathbb{Z}^2$ SFT whose one-dimensional projective subdynamics along a non-cardinal direction (principal diagonal) is non-sofic.

Example 5.7 (A block gluing $\mathbb{Z}^2$ SFT with a uniformly mixing non-sofic synchronized system as its projective subdynamics along a diagonal)**.** We will construct the corresponding $\mathbb{Z}^2$ SFT X_4^ω for this example in two steps. First let $\mathcal{A}_4 := \{0,1,2,3\}$ be the alphabet of a $\mathbb{Z}^2$ SFT $X_4 := \mathsf{X}(\mathcal{F}_4) \subsetneq \mathcal{A}_4^{\mathbb{Z}^2}$ defined by the following set of forbidden patterns:

$$\mathcal{F}_4 := \Big\{ 0\,2\,,\ 1\,2\,,\ 2\,1\,,\ 2\,3\,,\ 3\,1\,,\ 3\,3\,, \begin{smallmatrix}0\\1\end{smallmatrix}, \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}1\\3\end{smallmatrix}, \begin{smallmatrix}3\\3\end{smallmatrix}, \begin{smallmatrix} & 2\\0 & \end{smallmatrix}, \begin{smallmatrix} & 0\\1 & \end{smallmatrix}, \begin{smallmatrix} & 1\\2 & \end{smallmatrix},$$
$$\begin{smallmatrix}1 & \\ & 3\end{smallmatrix}, \begin{smallmatrix}2 & \\ & 3\end{smallmatrix}, \begin{smallmatrix}1 & \\ 0 & 1\end{smallmatrix}, \begin{smallmatrix}3 & \\ 0 & 3\end{smallmatrix}, \begin{smallmatrix}0 & \\ 3 & 0\end{smallmatrix}, \begin{smallmatrix}0 & 3\\ & 0\end{smallmatrix}, \begin{smallmatrix}2 & 0\\ & 2\end{smallmatrix}, \begin{smallmatrix}3 & 0\\ & 3\end{smallmatrix}, \begin{smallmatrix}1 & 1\\ 0 & 0\end{smallmatrix}, \begin{smallmatrix}1 & 3\\ 0 & 0\end{smallmatrix}, \begin{smallmatrix}2 & 0\\ 2 & 0\end{smallmatrix}, \begin{smallmatrix}2 & 0\\ 3 & 0\end{smallmatrix} \Big\}.$$

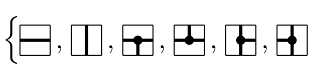
Observe the projective subdynamics $\mathsf{P}_L(X_4)$ of X_4 seen along the sublattice $L := \langle \vec{\mathbb{1}} \rangle_{\mathbb{Z}} \lneq \mathbb{Z}^2$ (principal diagonal). We claim it coincides with the topologically mixing non-sofic $\mathbb{Z}$ coded system $\mathsf{C}(\mathcal{C}_4) \subsetneq \mathcal{A}_4^{\mathbb{Z}}$ given by the (uniquely decipherable) non-regular, context-free code $\mathcal{C}_4 := \{0\,1^n\,2^n,\ 0\,1^n\,3\,2^n \mid n \in \mathbb{N}_0\}$. First the transitions ${}_2{}^3$, ${}_3{}^1$ and ${}_3{}^3$ are excluded implicitly in X_4 (those patterns can not be extended to any locally admissible 2×2 block). Moreover the forbidden patterns in $\mathcal{F}_4$ imply that the number of 1's and 2's in a block $0\,1^m\,2^n\,0$ resp. $0\,1^m\,3\,2^n\,0$ $(m,n \in \mathbb{N}_0)$ seen along the principal diagonal of a point in X_4 have to coincide by forcing a unique evolution in a (finite) triangular region to the NW of the diagonal (see Figure 4). Details of the argument are left to the reader. On the other hand any point $y \in \mathsf{C}(\mathcal{C}_4)$ can be put on the principal diagonal of a valid point $x \in X_4$ such that $x|_L = y$, $x|_{\vec{\imath}} = 0$ for all $\vec{\imath} = (\vec{\imath}_1, \vec{\imath}_2) \in \mathbb{Z}^2$ with $\vec{\imath}_1 > \vec{\imath}_2$ and all symbols $x_{\vec{\imath}}$ with $\vec{\imath}_1 < \vec{\imath}_2$ are either determined by the unique evolution of a subword of some codeword in $\mathcal{C}_4$ appearing in y or if not forced by this can be chosen to be 0.

```
⋱    2222222000000000000000000113000011111    ⋰
     2222220000003222222200000110000011111
     2222200000001322222000000100000011110
     2222000000001132220032200000000011100
     3220003220001113200013000000000011000
…    1300001300001111003010032222000010000    …
     1000001000001110000000013220000000030
     0000000000001100032000011300000000000
     0000000000001000010000011000000000000
     3222220000000000000000300100003222222222
⋰    1322200032222000000000000000132222222    ⋱
```

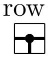
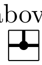
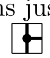
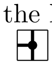
FIGURE 4. A finite rectangular pattern seen in a typical point in X_4.

As all future sets of the family $\{\mathsf{F}_{\mathsf{C}(\mathcal{C}_4)}(0^\infty\,1^n\,3)\}_{n \in \mathbb{N}_0}$ are distinct the $\mathbb{Z}$ shift $\mathsf{P}_L(X_4) = \mathsf{C}(\mathcal{C}_4)$ is not sofic. Using a similar argument as in Claim 5.1.2 to prove two arbitrary globally admissible words/patterns can be glued together as soon as they are separated by a distance linear in their length/diameter we can show that $\mathsf{P}_L(X_4)$ and in fact even X_4 is topologically mixing. However the L-projective subdynamics is not uniformly mixing and in particular X_4 is not block gluing. Hence we have to modify our construction to get the properties claimed above.

The technique used here to convert a non-block gluing shift into a block gluing one is quite general and involves introducing 6 new symbols as follows: Let

$\mathcal{A}_\omega := \Big\{$  $\Big\}$

be the so called *wire alphabet* (see [16]) consisting of square tiles containing either a piece of horizontal or vertical wire or a T-junction of branching wires. Now define a $\mathbb{Z}^2$ SFT $X_4^\omega \subsetneq (\mathcal{A}_4 \dot\cup \mathcal{A}_\omega)^{\mathbb{Z}^2}$ where the old symbols from $\mathcal{A}_4$ still have to respect the local rules defined by $\mathcal{F}_4$ as before while the new wire symbols from $\mathcal{A}_\omega$ are subject to horizontal and vertical adjacency rules requiring any piece of wire present in a tile at one coordinate to continue across the corresponding vertical/horizontal tile-edges hit by it into the neighboring wire tile. This way each wire has to continue forever – bifurcating into two branches at every one of the T-junction tiles – without starting or terminating at any finite coordinate.[8] In particular the symbols 0, 1, 2 and 3 from $\mathcal{A}_4$ may appear only next to a wire symbol from $\mathcal{A}_\omega$ if no *"wires have to continue"*-rule is violated, i.e. if the edge shared by the two symbols is not hit by a piece of wire. In consequence any valid point in X_4^ω is partitioned into a mosaic of (finite and/or infinite) rectangular regions being surrounded by but not containing any wire symbols. Each of those rectangular regions is itself filled by a locally admissible pattern from the previously constructed $\mathbb{Z}^2$ SFT X_4.

The proof that X_4^ω is block gluing follows exactly the argument given for the original wire shift in [16, Lemma 3.1] so we just mention the rough idea and refer to our earlier paper for details: Suppose we are given any pair of globally admissible patterns on (finite) solid blocks separated by a vertical gap of (at least) 2 rows. We surround each of those patterns with two infinite horizontal wires – one in the row just above, one in the row just below the pattern – where we use T-junctions  resp.  to feed all wires hitting the top resp. bottom of the given pattern. These two horizontal wires are then connected to each other using two vertical wires which run in the columns just to the left resp. right of our rectangular pattern, this time using T-junctions  resp.  to continue all wires hitting the pattern's left resp. right border. Now all *"wires have to continue"*-rules are satisfied and the remaining coordinates on the outside of the two patterns can be filled with symbols from $\mathcal{A}_4$ respecting the constraints given by $\mathcal{F}_4$. The construction for two rectangular patterns separated horizontally by a gap of at least 2 columns is symmetric.

Now it immediately follows that the L-projective subdynamics $\mathsf{P}_L(X_4^\omega)$ is uniformly mixing (Lemma 6.1) and clearly the symbol 0 as well as the wire symbols act as synchronizing words in $\mathsf{P}_L(X_4^\omega)$. Nevertheless $\mathsf{P}_L(X_4^\omega)$ is still not sofic, as again all future sets in the family $\left\{\mathsf{F}_{\mathsf{P}_L(X_4^\omega)}(0^\infty\, 1^n\, 3)\right\}_{n\in\mathbb{N}_0}$ are distinct.

6. Realizing mixing $\mathbb{Z}$ sofics as projective subdynamics in strongly irreducible $\mathbb{Z}^2$ SFTs

In this section we describe a technique to build a strongly irreducible $\mathbb{Z}^2$ SFT which realizes an arbitrary mixing $\mathbb{Z}$ sofic as its stable $\mathbb{Z}$-projective subdynamics. This construction automatically generalizes to strongly irreducible $\mathbb{Z}^d$ SFTs, thus proving Item (6) of our main theorem (Theorem 3.6). It can be seen as an optimal result as (uniform) mixingness of the $\mathbb{Z}$ shift showing up along the horizontal sublattice is an unavoidable restriction (Lemma 6.1) and soficness of the $\mathbb{Z}$-projective subdynamics was proven already under weaker conditions in Section 4.

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[8]Another way to look at the $\mathbb{Z}^2$ SFT X_4^ω is to think of it as the $\mathbb{Z}^2$ wire shift (introduced in [16]) with 4 types of blanks denoted by the symbols 0, 1, 2 and 3 which are subject to the additional adjacency rules specified in $\mathcal{F}_4$.

The following lemma shows that the one-dimensional projective subdynamics of uniformly mixing $\mathbb{Z}^d$ shifts again exhibit uniform mixing. This in particular excludes many of the subtleties found in the classification of sofic projective subdynamics of general $\mathbb{Z}^d$ SFTs in [13].

Lemma 6.1. *Let X be a block gluing $\mathbb{Z}^d$ shift and $L \lneq \mathbb{Z}^d$ a one-dimensional sublattice, then the L-projective subdynamics $\mathsf{P}_L(X)$ is a uniformly mixing $\mathbb{Z}$ subshift. Moreover the mixing distance of the subshift $\mathsf{P}_L(X)$ is bounded by the separation constant (gap size) of X.*

Proof. Given a one-dimensional sublattice $L = \langle \vec{v} \rangle_{\mathbb{Z}} \lneq \mathbb{Z}^d$ generated by an integer vector $\vec{v} \in \mathbb{Z}^d$ we choose a cardinal hyperplane $H_{\vec{e}_i} \subsetneq \mathbb{Z}^d$ perpendicular to a base vector $\vec{e}_i$ ($i \in \{1, 2, \ldots, d\}$) whose intersection with L is the singleton $\{\vec{0}\}$.

For any pair $y^{(1)}, y^{(2)} \in \mathsf{P}_L(X)$ choose points $x^{(1)}, x^{(2)} \in X$ so that $x^{(1)}|_L = y^{(1)}$ and $x^{(2)}|_L = y^{(2)}$. Denote by $g \in \mathbb{N}_0$ the (smallest) gap size for X. For any fixed $m \geq g$ block gluing of X allows us to construct a point $x \in X$ such that

$$x|_{H_{\vec{e}_i} - \mathbb{N}\vec{e}_i} = x^{(1)}|_{H_{\vec{e}_i} - \mathbb{N}\vec{e}_i} \qquad \text{and} \qquad x|_{H_{\vec{e}_i} + \mathbb{N}_0\vec{e}_i + m\vec{v}} = x^{(2)}|_{H_{\vec{e}_i} + \mathbb{N}_0\vec{e}_i} \; .$$

However restricting x to the sublattice L then gives a point $y \in \mathsf{P}_L(X)$ with $y|_{(-\infty,0)} = y^{(1)}|_{(-\infty,0)}$ and $y|_{[m,\infty)} = y^{(2)}|_{[0,\infty)}$.

This yields a way to construct globally admissible points in X realizing any pair $u := y^{(1)}|_{[-|u|,0)}, w := y^{(2)}|_{[0,|w|)} \in \mathcal{L}(\mathsf{P}_L(X))$ of words separated by a given distance $m \geq g$ and shows uniform mixingness of $\mathsf{P}_L(X)$ with a distance $M := g \in \mathbb{N}_0$. $\square$

Remark 6.2. Tunneled projectively fullness along L does not imply mixingness of the L-projective subdynamics. E.g. the $\mathbb{Z}^2$ SFT which is the full $\mathbb{Z}$-extension of the system consisting of two fixed points is $\mathbb{Z}$-TPF but its $\mathbb{Z}$-projective subdynamics, i.e. the two point system, lacks any kind of mixingness.

Recall from Section 2 that any $\mathbb{Z}$ sofic can be represented by a right-resolving graph G such that $S = \mathsf{S}(G)$ is the collection of all bi-infinite sequences of edge labels read along bi-infinite paths in G [12, Theorem 3.3.2]. We start the preparations for our Realization Theorem 6.4 showing the existence of certain $\mathbb{Z}$ subSFTs inside non-trivial mixing $\mathbb{Z}$ sofics. Those subSFTs will then be used in the construction of a strongly irreducible $\mathbb{Z}^2$ SFT realizing a given $\mathbb{Z}$ sofic projective subdynamics.

Lemma 6.3. *Let $S = \mathsf{S}(G)$ be any non-trivial mixing $\mathbb{Z}$ sofic presented by a right-resolving graph $G = (V_G, E_G, \lambda_G)$. Then for some $M \in \mathbb{N}$ large enough there exists a collection of $2|V_G| + 1$ distinct words $\mathcal{W}_{\text{unif}} := \{w_0, w_1, \ldots, w_{2|V_G|}\} \subseteq \mathcal{L}_M(S)$, all having the same length M, such that the subshift $Y_{\text{unif}} \subseteq S$ consisting of all bi-infinite concatenations of elements in $\mathcal{W}_{\text{unif}}$ is a $(3M-1)$-step $\mathbb{Z}$ SFT[9] inside S.*

In addition Y_{unif} (and thus S) contains a $(4M-1)$-step $\mathbb{Z}$ SFT $Y'_{\text{unif}} \subseteq Y_{\text{unif}}$ consisting of all points in Y_{unif} which avoid occurrences of the word w_0 in maximal runs of less than 3 (complete) repetitions and which also do not contain subwords of the form $w_0\, w_i$ with $i \not\equiv 0 \mod |V_G|$.

Finally there is a family of M distinct words $\mathcal{W}_{\text{fill}} := \{w_0^{(m)} \mid 0 \leq m < M\} \subseteq \mathcal{L}(S)$ of varying lengths $|w_0^{(m)}| = \widetilde{M} + m$ (for $\widetilde{M} \in \mathbb{N}$ a constant with $\widetilde{M} > 9M$), all of which start and end in $w_0\, w_0\, w_0$ and such that the subshift $Y_{\text{fill}} \subseteq S$ consisting of

[9]A $\mathbb{Z}$ SFT is $(N-1)$-*step* if it can be defined by a family of forbidden words all of which have length (at most) N. Equivalently all admissible words of length $N-1$ are synchronizing.

all bi-infinite concatenations of elements in $\mathcal{W}_{\text{fill}} \,\dot\cup\, \{w_0\}$ *is a mixing* $(\widetilde{M}+7M-2)$*-step* $\mathbb{Z}$ *SFT inside* S*.*

Proof. The construction of the collection $\mathcal{W}_{\text{unif}}$ is similar to the one in Lemma 4.1 in [13] but in order for the presentation to be self contained we repeat the slightly modified argument. As S is non-trivial and mixing it has positive entropy. This then guarantees the existence of two words $t_1 := \lambda_G(c_1),\ t_2 := \lambda_G(c_2) \in \mathcal{L}(S)$ that label cycles c_1, c_2 both beginning and ending at the same vertex $v^* \in V_G$, i.e. $v^* = \mathfrak{i}_G(c_1) = \mathfrak{t}_G(c_1) = \mathfrak{i}_G(c_2) = \mathfrak{t}_G(c_2)$ and such that t_1 and t_2 differ in their first symbol. Moreover, since S is mixing we may assume that the graph G is strongly connected and aperiodic, i.e. the greatest common divisor of its cycle lengths is 1. Thus w.l.o.g. we can choose the two (possibly non-primitive) cycles such that the length of c_1 equals the length of c_2 plus 1 (hence $|t_1| = |t_2| + 1$ as well).

Let $u_1 := t_1\, t_2\, t_2\, t_1$ and $u_2 := t_1\, t_2\, t_1\, t_2$, then $u_1, u_2 \in \mathcal{L}_{2|t_1\, t_2|}(S)$ (t_1, t_2 both label cycles at the vertex v^*) and considering that $(u_2)^\infty$ has period $|t_1\, t_2|$ but $(u_1)^\infty$ does not, it is clear that u_1 can not appear as a subword of $u_2\, u_2$. This allows us to define the set of words $\mathcal{W}_{\text{unif}} := \{w_0, w_1, \ldots, w_{2|V_G|}\} \subseteq \mathcal{L}_M(S)$ where $M := 8(|V_G|+1)\,|t_1\, t_2| \geq 16\,|t_1\, t_2|$ and $w_n := u_1\,(u_2)^{2|V_G|+n+2}\,u_1\,(u_2)^{2|V_G|-n}$ for each $0 \leq n \leq 2\,|V_G|$. It is easy to see that the $\mathbb{Z}$ shift Y_{unif} consisting of all bi-infinite concatenations of words from $\mathcal{W}_{\text{unif}}$ is actually a subshift of S; all words in $\mathcal{W}_{\text{unif}}$ start and end at v^* thus can be concatenated freely. In addition we can prove that $Y_{\text{unif}} \subseteq S$ is a $(3M-1)$-step $\mathbb{Z}$ SFT by showing that any word $\widetilde{w} \in \mathcal{L}_{3M-1}(Y_{\text{unif}})$ has exactly one decomposition into subwords $\widetilde{w} = p\, w\, w'\, q$ with unique $w, w' \in \mathcal{W}_{\text{unif}}$ and $p, q \in \mathcal{L}(Y_{\text{unif}})$ a possibly empty prefix resp. suffix both of length strictly less than M; for details of this argument see [13, Lemma 4.1]. Hence each $\widetilde{w}$ is synchronizing and Y_{unif} is a $(3M-1)$-step $\mathbb{Z}$ SFT as claimed.

To get $Y'_{\text{unif}} \subseteq S$ we modify the $\mathbb{Z}$ SFT Y_{unif} just constructed by additionally excluding all words of the form $w_i\, w_0\, w_j$ and $w_i\, w_0\, w_0\, w_j$ for $0 < i, j \leq 2\,|V_G|$, thus allowing the word w_0 to appear only in blocks formed by at least three repetitions and also excluding all words of the form $w_0\, w_i$ where $1 \leq i \leq |V_G|-1$ or $|V_G|+1 \leq i \leq 2\,|V_G|-1$, thus allowing the word w_0 only to be followed by w_i with $i \equiv 0 \mod |V_G|$. Clearly those restrictions are given by local rules – the maximal length of a newly excluded word is $4M$ – and so the modified subshift $Y'_{\text{unif}} \subseteq Y_{\text{unif}}$ respecting those additional rules still is a $(4M-1)$$=\max\{3M-1, 4M-1\}$-step $\mathbb{Z}$ SFT.

Furthermore let $\widetilde{M} := 7M + 2(M+2)\,|t_2|$ and for $0 \leq m < M$ define words

$$w_0^{(m)} := w_0\, w_0\, w_0\, t_1^{M+m}\, t_2^{M-m+4}\, w_0\, w_0\, w_0$$

of lengths $\left|w_0^{(m)}\right| = 6M + (M+m)\,|t_1| + (M-m+4)\,|t_2| = \widetilde{M} + m$ covering the interval $\left[\widetilde{M}, \widetilde{M}+M\right) \cap \mathbb{Z}$ (of length M). Those words then form another collection $\mathcal{W}_{\text{fill}} := \left\{w_0^{(m)} \;\middle|\; 0 \leq m < M\right\} \subseteq \mathcal{L}(S)$. Note that again each word $w_0^{(m)}$ starts and ends at v^* and therefore those words can be freely concatenated with each other and with the words from $\mathcal{W}_{\text{unif}}$ still staying inside S. We form the subshift $Y_{\text{fill}} \subseteq S$ consisting of all bi-infinite concatenations of words from the set $\mathcal{W}_{\text{fill}} \,\dot\cup\, \{w_0\} \subseteq \mathcal{L}(S)$ which again is a $\mathbb{Z}$ SFT, this time a $(\widetilde{M}+7M-2)$-step $\mathbb{Z}$ SFT. This easily follows from the fact that the word $(w_0)^6$ (six times concatenating w_0 with itself) is synchronizing, can not overlap non-trivially with itself nor with any word in $\mathcal{W}_{\text{fill}}$ and appears in every point in Y_{fill} syndetically with maximal gap size $\widetilde{M}+M-1$. Y_{fill} is mixing due to the varying lengths of words in $\mathcal{W}_{\text{fill}}$. □

Claim 6.3.1. In the situation of Lemma 6.3, the only element of $\mathcal{W}_{\text{unif}}$ that appears in the language $\mathcal{L}(Y_{\text{fill}})$ is the word w_0. Moreover inside any word $w_0^{(m)}$ an occurrence of w_0 is only possible in the prefix (resp. suffix) $w_0\, w_0\, w_0$ where w_0 has to be properly aligned (i.e. it has to start at coordinate 1, $M+1$ or $2M+1$ resp. $\left|w_0^{(m)}\right| - 3M + 1$, $\left|w_0^{(m)}\right| - 2M + 1$ or $\left|w_0^{(m)}\right| - M + 1$).

Proof. The key to show this is to observe that certain words are not subwords of others. Using periodicity we already mentioned that $u_1 \not\sqsubseteq u_2\, u_2$. Similarly $t_1\, t_2 \not\sqsubseteq t_1\, t_1\, t_1$ (in fact not even its prefix of length $|t_1| + 1$ can show up), $t_1\, t_1 \not\sqsubseteq t_2\, t_2\, t_2\, t_2$ and $t_2\, t_1 \not\sqsubseteq t_2\, t_2\, t_2$ (again even the prefix of length $|t_2| + 1$ does not appear). Now to prove the claim we can restrict ourselves to determining which elements of $\mathcal{W}_{\text{unif}}$ show up as subwords of some $w_0^{(m)} \in \mathcal{W}_{\text{fill}}$ $(0 \le m < M)$; simply note that due to the prefix (suffix) $w_0\, w_0\, w_0$ all words in $\mathcal{L}_M(Y_{\text{fill}})$ already appear as subwords of some element in $\mathcal{W}_{\text{fill}}$. So let us suppose $w \in \mathcal{W}_{\text{unif}} \cap \mathcal{L}_M(Y_{\text{fill}})$ is such an element. First we exclude any appearance of w inside the central piece $t_1^{M+m}\, t_2^{M-m+4}$ of $w_0^{(m)}$ using that w starts with $u_1 = t_1\, t_2\, t_2\, t_1$ whose prefix of length $|t_1| + 1$ is not allowed inside t_1^{M+m} and whose suffix $t_2\, t_1$ is not allowed in t_2^{M-m+4}. In particular this shows there is no way to cover any of the words $t_1^{M+m}\, t_2^{M-m+4}$ $(0 \le m < M)$ with a concatenation of elements from $\mathcal{W}_{\text{unif}}$ and thus none of them is contained in $\mathcal{L}(Y_{\text{unif}})$. Moreover w can not be seen as a subword of $w_0\, t_1^{M+m}$ starting more than $|u_1|$ symbols from its left end either, as otherwise a suffix of w of length more than $|u_1|$ thus containing the subword $t_1\, t_2$ would be contained inside t_1^{M+m} which is not possible. This only leaves the possibility of having w overlapping with $w_0^{(m)}$'s prefix or suffix $(w_0)^3$ for at least half of its length M. Following from the argument used to show Y_{unif} is a $(3M-1)$-step $\mathbb{Z}$ SFT described above (see also [13, Lemma 4.1]) this is excluded unless w is exactly aligned, i.e. w has to coincide with one of the copies of w_0 as claimed. □

Theorem 6.4. *For any mixing $\mathbb{Z}$ sofic S there exists a strongly irreducible $\mathbb{Z}^2$ SFT X which realizes S as its stable $\mathbb{Z}$-projective subdynamics such that $\mathsf{P}_{\mathbb{Z}}(X) = S$.*

Moreover the subshift X can be forced to have the property that any configuration on $\mathbb{Z}^{[1]} = \mathbb{Z} \times \{-1, 0, 1\}$ which is locally admissible for X contains a point of S in its central row, i.e. $S = \mathsf{P}_{\mathbb{Z}}(X) = X_{\mathbb{Z},1}$ is the 1-step stable $\mathbb{Z}$-projective subdynamics of such a strongly irreducible $\mathbb{Z}^2$ SFT X.

Proof. Note that the trivial $\mathbb{Z}$ sofic $S = \left\{a^{\mathbb{Z}}\right\}$ is the projective subdynamics of the strongly irreducible (trivial) $\mathbb{Z}^2$ SFT $X = \left\{a^{\mathbb{Z}^2}\right\}$. Hence we may assume our $\mathbb{Z}$ sofic $S \subseteq \mathcal{A}^{\mathbb{Z}}$ to be non-trivial and thus apply Lemma 6.3 to find the two $\mathbb{Z}$ SFTs $Y_{\text{fill}} \subseteq S$ and $Y'_{\text{unif}} \subseteq S$. This in particular gives us two large enough sets of words $\mathcal{W}_{\text{fill}}, \mathcal{W}_{\text{unif}} \subsetneq \mathcal{L}(S)$ which can be freely concatenated without violating any rules of S while being controlled by SFT-rules only. As we will see, the role of the words in $\mathcal{W}_{\text{fill}}$ having lengths covering the interval $[\widetilde{M}, \widetilde{M} + M)$ is to fill long enough gaps making X strongly irreducible while words in $\mathcal{W}_{\text{unif}}$ having uniform length M act as marker blocks used to code vertices of the (right-resolving) graph presentation $G = (V_G, E_G, \lambda_G)$ of our $\mathbb{Z}$ sofic $S = \mathsf{S}(G)$ as has been used in Lemma 6.3.

To construct a strongly irreducible $\mathbb{Z}^2$ SFT $X \subseteq \mathcal{A}^{\mathbb{Z}^2}$ with $\mathsf{P}_{\mathbb{Z}}(X) = S$ we will employ local rules to force every point in X to look mostly like a point in the full-$\mathbb{Z}$-extension of the mixing $\mathbb{Z}$ SFT Y_{fill} with possibly some (finite or infinite) regions – called *defects* – of non-Y_{fill}-patterns interspersed. It is the existence of those defects which lets us realize arbitrary (subwords of) points in S in the central row of a defect. A defect always fills an aligned portion of three consecutive rows, i.e. defects have a rectangular shape of height 3 but arbitrary (finite or infinite) length. Moreover each defect will be surrounded by an admissible pattern of ${Y_{\text{fill}}}^{\mathbb{Z}}$ filling a rectangular frame extending at least three rows above and below the defect and at least $\widetilde{M} + 10M$ coordinates to its left and right. Thus any defect will always be adequately separated from possibly existing other defects as shown in Figure 5. Further local constraints then enforce that the upper and lower row – called the *checker rows* – of a defect both have to contain the same (in)finite subword which has to be admissible for the $\mathbb{Z}$ SFT Y'_{unif}, i.e. both rows contain a concatenation of words from $\mathcal{W}_{\text{unif}}$. To allow transitioning from and to the surrounding Y_{fill}-patterns at the left/right end of a defect the concatenations of $\mathcal{W}_{\text{unif}}$ blocks in both checker rows will start with and end in at least three occurrences of the word $w_0 \in \mathcal{W}_{\text{unif}}$. The row in the middle – called the *tester row* – of a defect may a priori contain an arbitrary (in)finite word over the alphabet $\mathcal{A}$, again with the restriction of having a prefix and suffix of $w_0\, w_0\, w_0$ to transition into the surrounding Y_{fill}-frame. However a final local rule will constrain each tester row to only contain a legal (in)finite subword of S, thus producing the desired $\mathbb{Z}$-projective subdynamics (see Figure 5).

To assure that a tester row is filled with an admissible subword of S we apply a local checking procedure similar to what was done in [13, Theorem 4.2]: A (finite or infinite) subword over $\mathcal{A}$ is admissible for S if and only if it corresponds to a valid walk along the labeled edges of the graph G. Hence one way to assure global

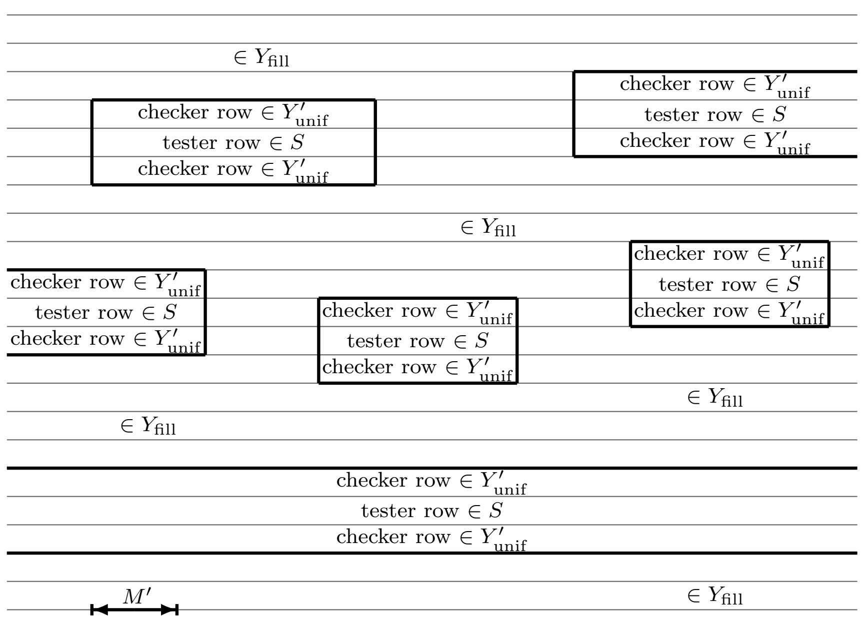

Figure 5. A typical point in X consisting of (finite and infinite) defect zones (marked by thicklines) – each one consisting of a tester row surrounded by its pair of checker rows – interspersed in a "sea" of Y_{fill} background.

validity of a long sequence seen in a tester row is to think of it as coming from a walk consisting of a concatenation of subpaths of length M (whose validity can be checked locally) while keeping track of the vertices visited at the junction of each pair of adjacent paths. We will code those vertices using the elements from the set $\mathcal{W}_{\text{unif}} \setminus \{w_0\}$. (For technical reasons each vertex in $V_G = \{v_1, v_2, \ldots, v_{|V_G|}\}$ can be coded by two distinct words $\mathcal{W}_{\text{unif}} \setminus \{w_0\} = \{w_1, w_2, \ldots, w_{2|V_G|}\}$.) In fact, as the two checker rows in a defect have to be filled with a concatenation of words from $\mathcal{W}_{\text{unif}}$, this naturally partitions the sequence seen in the defect's tester row into blocks of length M. A local rule can then check whether each such block corresponds to the label of a valid path (of length M) in G whose initial vertex is coded by the checker-row-word sitting immediately above (and below) it and whose terminal vertex is coded by the next checker-row-word sitting to the right (which thus automatically equals the initial vertex of the following subpath). Hence the global structure of the sequence in the tester row is checked by using only local information which then forces it to be an admissible (in)finite word of S as claimed.

Now we are ready to formally define the $\mathbb{Z}^2$ SFT $X \subseteq \mathcal{A}^{\mathbb{Z}^2}$, which lives on the same alphabet $\mathcal{A}$ as S, by specifying the following local rules (let $M' := \widetilde{M} + 11M$):

(R1) *Forcing the general setup of defects and surrounding Y_{fill} patterns:*
For a pattern $P \in \mathcal{A}^{[1,M'] \times [1,11]}$ such that the word $P|_{[1,M'] \times \{6\}} \notin \mathcal{L}(Y_{\text{fill}})$ there are three possibilities: Either
- $P|_{[1,M'] \times \{4\}} = P|_{[1,M'] \times \{6\}} \in \mathcal{L}(Y'_{\text{unif}})$ and for all $j \in \{1,2,3,7,8,9\}$ we have $P|_{[1,M'] \times \{j\}} \in \mathcal{L}(Y_{\text{fill}})$ (i.e. defect sits in rows $4,5,6$) or
- $P|_{[1,M'] \times \{5\}} = P|_{[1,M'] \times \{7\}} \in \mathcal{L}(Y'_{\text{unif}})$ and for all $j \in \{2,3,4,8,9,10\}$ we have $P|_{[1,M'] \times \{j\}} \in \mathcal{L}(Y_{\text{fill}})$ (i.e. defect sits in rows $5,6,7$) or
- $P|_{[1,M'] \times \{6\}} = P|_{[1,M'] \times \{8\}} \in \mathcal{L}(Y'_{\text{unif}})$ and for all $j \in \{3,4,5,9,10,11\}$ we have $P|_{[1,M'] \times \{j\}} \in \mathcal{L}(Y_{\text{fill}})$ (i.e. defect sits in rows $6,7,8$).

(R2) *Forcing vertex marker setup along the top and bottom row of defects:*
A pattern $P \in \mathcal{A}^{[1,M'] \times [1,3]}$ for which either $P|_{[1,M'] \times \{1\}}, P|_{[1,M'] \times \{3\}} \notin \mathcal{L}(Y_{\text{fill}})$ or $P|_{[1,M'] \times \{2\}} \notin \mathcal{L}(Y'_{\text{unif}}) \cup \mathcal{L}(Y_{\text{fill}})$ has to satisfy $P|_{[1,M'] \times \{1\}} = P|_{[1,M'] \times \{3\}} \in \mathcal{L}(Y'_{\text{unif}})$ (i.e. the entire P is contained in a defect).
Moreover for each $i \in [1, M' - 2M + 2]$ let $i^* \in [i, i+M)$ be the unique coordinate for which $P|_{[i^*, i^*+M) \times \{1\}} \in \mathcal{W}_{\text{unif}}$, where we exact the following: If $P|_{[i^*, i^*+M) \times \{1\}} = w_0$ then we require $P|_{[i^*, i^*+M) \times \{2\}} = w_0$ as well, while $P|_{[i^*, i^*+M) \times \{1\}} \neq w_0$ forces $P|_{[i^*, i^*+M) \times \{2\}} \neq P|_{[i^*, i^*+M) \times \{1\}}$.

(R3) *Implementing the local checking procedure for the central row inside a defect:*
A pattern $P \in \mathcal{A}^{[1,3M] \times [1,3]}$ with $P|_{[1,3M] \times \{1\}} = P|_{[1,3M] \times \{3\}} \in \mathcal{L}(Y'_{\text{unif}})$ and $P|_{[M+1,2M] \times \{1\}} \in \mathcal{W}_{\text{unif}} \setminus \{w_0\}$ enforces two additional conditions:
First of all $P|_{[iM+1, iM+M] \times \{1\}} \in \mathcal{W}_{\text{unif}}$ for all $i \in \{0,1,2\}$ (this is automatic) and for each $i \in \{0,1,2\}$ whenever $P|_{[iM+1, iM+M] \times \{1\}} = w_0$ then $P|_{[iM+1, iM+M] \times \{2\}} = w_0$ as well, whereas $P|_{[iM+1, iM+M] \times \{1\}} \neq w_0$ implies $P|_{[iM+1, iM+M] \times \{2\}} \neq P|_{[iM+1, iM+M] \times \{1\}}$.
Secondly suppose that $P|_{[1,M] \times \{1\}} = w_n$, $P|_{[M+1,2M] \times \{1\}} = w_{n'}$ and $P|_{[2M+1,3M] \times \{1\}} = w_{n''}$ for $n, n', n'' \in \{0, 1, \ldots, 2|V_G|\}$. Then the word $P|_{[1,M] \times \{2\}}$ has to be the label of a valid path (of length M) from vertex $v_{(n \mod |V_G|)+1}$ to vertex $v_{(n' \mod |V_G|)+1}$ and the word $P|_{[M+1,2M] \times \{2\}}$ has to be the label of a valid path (of length M) from vertex $v_{(n' \mod |V_G|)+1}$ to vertex $v_{(n'' \mod |V_G|)+1}$. (W.l.o.g. assume v_1 to be the vertex, denoted v^* in the proof of Lemma 6.3, at which all words in $\mathcal{W}_{\text{unif}}$ start and end.)

Next we show those three local rules already force $\mathsf{P}_{\mathbb{Z}}(X) = S$ by proving the mutual containment of those two $\mathbb{Z}$ subshifts:

Claim 6.4.1 ($S \subseteq \mathsf{P}_{\mathbb{Z}}(X)$)**.** Given any point $s \in S$ there is an easy way to construct a point $x \in X$ such that $x|_{\mathbb{Z}\times\{0\}} = s$: Denote by $y_{\text{fill}} := (w_0)^\infty \in Y_{\text{fill}}$ the periodic point formed from concatenating copies of the word $w_0 \in \mathcal{W}_{\text{unif}} \cap \mathcal{L}(Y_{\text{fill}})$ and let $(e_i \in E_G)_{i\in\mathbb{Z}}$ be a bi-infinite path in G representing s, i.e. $s = \big(\lambda_G(e_i)\big)_{i\in\mathbb{Z}}$. Concatenating only words from $\mathcal{W}_{\text{unif}} \setminus \{w_0\}$ we then form a point $y_{\text{check}} \in Y'_{\text{unif}}$ such that $y_{\text{check}}|_{[iM,(i+1)M)} \in \big\{w_n, w_{n+|V_G|} \;\big|\; 1 \le n \le |V_G| \;\wedge\; \mathfrak{i}_G(e_{iM}) = v_n\big\}$ with $y_{\text{check}}|_{[iM,(i+1)M)} \neq s|_{[iM,(i+1)M)}$ for all $i \in \mathbb{Z}$. Defining $x|_{\mathbb{Z}\times\{0\}} := s$, $x|_{\mathbb{Z}\times\{\pm1\}} := y_{\text{check}}$ and $x|_{\mathbb{Z}\times\{j\}} := y_{\text{fill}}$ for all $j \in \mathbb{Z}\setminus\{0,\pm1\}$ now obviously satisfies all local rules (R1), (R2) and (R3). Rows $-1, 0, 1$ form a bi-infinite defect with tester row s and checker rows y_{check} inside a "sea" of $y_{\text{fill}} \in Y_{\text{fill}}$ background; x therefore yields an admissible point in X realizing s along its horizontal sublattice.

Claim 6.4.2 ($\mathsf{P}_{\mathbb{Z}}(X) \subseteq S$)**.** Suppose for a contradiction that there exists a point $x \in X$ with $x|_{\mathbb{Z}\times\{0\}} \notin S$. Hence in particular $x|_{\mathbb{Z}\times\{0\}} \notin Y'_{\text{unif}} \cup Y_{\text{fill}}$ and as $Y'_{\text{unif}}, Y_{\text{fill}}$ both are $(M'-4M-2)$-step $\mathbb{Z}$ SFTs there has to be some $a \in \mathbb{Z}$ such that the word $x|_{[a+2M,a+M'-2M)\times\{0\}} \notin \mathcal{L}(Y'_{\text{unif}}) \cup \mathcal{L}(Y_{\text{fill}})$. W.l.o.g. we may assume $a = 1$ which by applying Rule (R2) would require $x|_{[1,M']\times\{-1\}} = x|_{[1,M']\times\{1\}} \in \mathcal{L}(Y'_{\text{unif}})$ and $x|_{[1,M']\times\{j\}} \in \mathcal{L}(Y_{\text{fill}})$ for all $j \in \{\pm2,\pm3,\pm4\}$ by Rule (R1); this means that there is a defect in rows $-1, 0, 1$. Recall from the proof of Lemma 6.3 that every Y'_{unif}-word of length at least $3M-1$ has a unique decomposition into blocks of $\mathcal{W}_{\text{unif}}$ and so let $1 \le a^* \le M$ be the sole coordinate for which $x|_{[a^*+iM,a^*+(i+1)M)\times\{\pm1\}} \in \mathcal{W}_{\text{unif}}$ for each $i \in \{0,1,\dots,I-1\}$ with $I := \Big\lfloor \frac{M'-a^*+1}{M} \Big\rfloor$. To simplify notation define the arithmetic progression of coordinates $(a_i := a^* + i\cdot M)_{i\in\mathbb{Z}}$.

Now $x|_{[a_1,a_{I-1})\times\{\pm1\}}$ can not be a pure concatenation of copies of w_0 as otherwise $x|_{[a_1,a_{I-1})\times\{0\}} = x|_{[a_1,a_{I-1})\times\{\pm1\}} = (w_0)^{I-2}$ (by the second part of Rule (R2)) would force $x|_{[1+2M,M'-2M]\times\{0\}} \in \mathcal{L}(Y_{\text{fill}})$ contradicting our initial assumption. This guarantees the existence of at least one $i \in \{1,2,\dots,I-2\}$ such that $x|_{[a_i,a_{i+1})\times\{\pm1\}} \in \mathcal{W}_{\text{unif}} \setminus \{w_0\}$. But then $x|_{[a_i,a_i+M')\times\{1\}}, x|_{[a_i,a_i+M')\times\{-1\}} \notin \mathcal{L}(Y_{\text{fill}})$ (recall that $\mathcal{W}_{\text{unif}} \cap \mathcal{L}(Y_{\text{fill}}) = \{w_0\}$ by Claim 6.3.1) and Rule (R2) would force these two words to coincide and to be elements of $\mathcal{L}(Y'_{\text{unif}})$. The same argument holds for the pair $x|_{[a_{i+1}-M',a_{i+1})\times\{1\}}$ and $x|_{[a_{i+1}-M',a_{i+1})\times\{-1\}}$ and in fact this situation would spread to the right and left until we reach an occurrence of (long enough concatenations of) the word w_0. There are four possible scenarios, all of which contradict our assumption $x|_{\mathbb{Z}\times\{0\}} \notin S$ as we will show now:

- If no occurrence of the word w_0 is ever seen in $x|_{\mathbb{Z}\times\{1\}}$ (nor in $x|_{\mathbb{Z}\times\{-1\}}$) then $x|_{\mathbb{Z}\times\{1\}} = x|_{\mathbb{Z}\times\{-1\}} \in Y'_{\text{unif}}$ and the second condition in Rule (R3) guarantees that actually $x|_{\mathbb{Z}\times\{0\}} \in S$: For every $i' \in \mathbb{Z}$ the subword $x|_{[a_{i'},a_{i'+1})\times\{0\}}$ has to be the label of a valid path of length M in G starting at vertex v_n and ending at vertex $v_{n'}$ where $x|_{[a_{i'},a_{i'+1})\times\{\pm1\}} \in \big\{w_n, w_{n+|V_G|}\big\} \subseteq \mathcal{W}_{\text{unif}}\setminus\{w_0\}$ and $x|_{[a_{i'+1},a_{i'+2})\times\{\pm1\}} \in \big\{w_{n'}, w_{n'+|V_G|}\big\} \subseteq \mathcal{W}_{\text{unif}} \setminus \{w_0\}$ determine the indices $n, n' \in \{1,2,\dots,|V_G|\}$. Overall this yields the label of a bi-infinite path in G, i.e. row $x|_{\mathbb{Z}\times\{0\}}$ contains a valid point in S.
- Suppose we do see a first occurrence of w_0 in $x|_{\mathbb{Z}\times\{1\}}$ (thus also in $x|_{\mathbb{Z}\times\{-1\}}$) to the left of $a_i = a^* + iM$, say at coordinates $[a_{i'}, a_{i'+1})$ with $i' < i$, but never to its right. Rule (R3) applied to pattern $x|_{[a_{i'},a_{i'+3})\times\{-1,0,1\}}$ then

implies that $x|_{[a_{i'},a_{i'+1})\times\{0\}} = w_0$ and the word immediately to its right, i.e. $x|_{[a_{i'+1},a_{i'+2})\times\{0\}}$, has to be the label of a valid path of length M in G which starts at vertex $v_{n_{i'+1}} = v_1$ and ends at some vertex $v_{n_{i'+2}} \in V_G$ uniquely determined by the word $x|_{[a_{i'+2},a_{i'+3})\times\{\pm1\}} \in \mathcal{W}_{\text{unif}} \setminus \{w_0\}$. Note that by definition of Y'_{unif} we have $x|_{[a_{i'+1},a_{i'+2})\times\{\pm1\}} \in \{w_{|V_G|}, w_{2|V_G|}\}$ which codes vertex v_1. Also note that by choice of v_1 being the initial and terminal vertex of the path used to define the word w_0, the condition of Rule (R3) on $x|_{[a_{i'},a_{i'+1})\times\{0\}} = w_0$ is satisfied as well. Successively moving M coordinates to the right always imposing Rule (R3) this situation persists; i.e. for all $i \in \mathbb{N}$ each word $x|_{[a_{i'+i},a_{i'+i+1})\times\{0\}}$ coincides with the label of a valid path of length M in G starting at vertex $v_{n_{i'+i}} \in V_G$ given by $x|_{[a_{i'+i},a_{i'+i+1})\times\{\pm1\}} \in \mathcal{W}_{\text{unif}} \setminus \{w_0\}$ and ending at vertex $v_{n_{i'+i+1}} \in V_G$ specified by $x|_{[a_{i'+i+1},a_{i'+i+2})\times\{\pm1\}} \in \mathcal{W}_{\text{unif}} \setminus \{w_0\}$. Thus the whole right-infinite sequence $x|_{[a_{i'},\infty)\times\{0\}}$ corresponds to the label of a right-infinite ray in G starting at vertex v_1.

- Similarly suppose we do see a first occurrence of w_0 to the right of $a_i = a^* + iM$ say at coordinates $[a_{i'}, a_{i'+1})$ with $i' > i$ but never to its left. A symmetric argument then shows that $x|_{[a_{i'},a_{i'+1})\times\{0\}} = w_0$ and the complete left-infinite sequence $x|_{(-\infty,a_{i'+1})\times\{0\}}$ corresponds to the label of a left-infinite ray in G ending at vertex v_1.
- Finally suppose we do see occurrences of w_0 both to the left and to the right of $a_i = a^* + iM$ (say at coordinates $[a_{i'}, a_{i'+1})$ for some $i' < i$ and at $[a_{i''}, a_{i''+1})$ for some $i'' > i$). Combining the arguments from the previous two cases assures that the finite word $x|_{[a_{i'},a_{i''+1})\times\{0\}}$ which starts and ends in a subword w_0 is the label of a finite path of length $(i'' - i' + 1)M$ in G starting and ending at vertex v_1.

Bringing to bear all 4 cases we have actually shown that the entire (in)finite tester row of any defect can be partitioned into finite and/or infinite pieces each containing an admissible subword of S starting and ending in a concatenation of copies of the word w_0. Moreover the vertex markers seen in the defect's checker rows guarantee the existence of corresponding (in)finite paths in G starting and ending at the vertex v_1 whose labels coincide with the subwords seen in those pieces of the tester row between any two occurrences of the block w_0. Hence we proved that words in $\mathcal{A}^{M'} \setminus \big(\mathcal{L}(Y'_{\text{unif}}) \cup \mathcal{L}(Y_{\text{fill}})\big)$ which are locally admissible in X or even just in $X_{\mathbb{Z},1}$ force a (possibly infinite) extension to both sides along the horizontal direction until a copy of w_0 is reached and those extensions have to be labels of valid paths in G. Thus we are left with the pieces of the sequence $x|_{\mathbb{Z}\times\{0\}}$ which are not part of any defect's tester row. Rule (R2) however forces all those to be admissible subwords of either Y'_{unif} or Y_{fill}, again given as labels of paths starting and ending at vertex v_1. Therefore the entire bi-infinite sequence $x|_{\mathbb{Z}\times\{0\}}$ is a concatenation of labels of those paths, i.e. $x|_{\mathbb{Z}\times\{0\}} \in S$ and the proof actually shows $\mathsf{P}_{\mathbb{Z}}(X) = X_{\mathbb{Z},1} = S$.

Finally we prove the claimed uniform mixingness of X:

Claim 6.4.3 (X is a strongly irreducible $\mathbb{Z}^2$ SFT)**.** Let $w^* := (y_{\text{fill}})^{\mathbb{Z}} \in X$ be the point with $y_{\text{fill}} := (w_0)^\infty \in Y_{\text{fill}}$ in each of its rows. In order to show strong irreducibility of X it suffices to find a constant $M^* \in \mathbb{N}$ such that any globally admissible pattern $P \in \mathcal{L}_F(X)$ on a finite shape $F \subsetneq \mathbb{Z}^2$ can be inserted into the "background" of aligned copies of w_0 given by w^* by changing this periodic $\mathbb{Z}^2$

configuration only in a neighborhood of F of fixed width M^*. More precisely for any choice of F and P there should always exist a point $x \in X$ with $x|_F = P$ and $x|_{\mathbb{Z}^2 \setminus (F + C_{M^*})} = w^*|_{\mathbb{Z}^2 \setminus (F + C_{M^*})}$ where $C_{M^*} := [-M^* \vec{\mathbb{1}}, M^* \vec{\mathbb{1}}] \subsetneq \mathbb{Z}^2$. Hence x sees P while keeping the aligned grid of w_0's except in a bounded margin around P. Now consider any pair of non-empty finite shapes $U, W \subsetneq \mathbb{Z}^2$ separated by at least $\delta_\infty(U, W) > 2M^* + M'$ and let $y, z \in X$ be arbitrary points. As a consequence of the embeddability of finite patterns, i.e. in particular $y|_U$ and $z|_W$, into w^* there exists a valid point $x \in X$ with $x|_U = y|_U$, $x|_W = z|_W$ and $x|_{\mathbb{Z}^2 \setminus (U + C_{M^*} \dot\cup W + C_{M^*})} = w^*|_{\mathbb{Z}^2 \setminus (U + C_{M^*} \dot\cup W + C_{M^*})}$. (Here we used that X is defined by local rules of size M'.) Hence $g := 2M^* + M'$ yields a gap size proving X to be strongly irreducible.

So we are left to find $M^* \in \mathbb{N}$. Recall that S is a mixing $\mathbb{Z}$ sofic presented on the graph G; thus we may assume G to be strongly connected and aperiodic. Moreover there exists a cycle c of length M and labeled $w_0 \in \mathcal{W}_{\text{unif}}$ which starts and ends at the vertex $v_1 \in V_G$. Let $t \in M\mathbb{N}_0$ be the smallest transition length being a multiple of the block length M necessary to connect this special vertex v_1 to an arbitrary vertex $v \in V_G$ as well as any vertex $v \in V_G$ back to v_1. It should be clear that concatenating with the cycle c we then have paths connecting v_1 to any other vertex (and any other vertex back to v_1) of all lengths $t + M\mathbb{N}_0$. Now a globally admissible pattern $P \in \mathcal{L}_F(X)$ might occur in some point $x^{(P)} \in X$, i.e. $x^{(P)}|_F = P$, which contains some defects intersecting both F and its complement $\mathbb{Z}^2 \setminus F$ (or lying entirely in the complement $\mathbb{Z}^2 \setminus F$). We will show how to locally modify such a point $x^{(P)}$ getting rid of all defects completely contained in the complement $\mathbb{Z}^2 \setminus F$ and shortening all defects intersecting F to prevent them from extending more than $t + M$ coordinates horizontally beyond the boundary of F as follows:

Suppose a defect extends far beyond the boundary of F to the right without ever re-entering F. Since its two checker rows contain the same concatenation of blocks from $\mathcal{W}_{\text{unif}}$ we can pick the horizontal coordinate outside F where in both checker rows the first of those blocks not intersecting F starts. Clearly this coordinate is at most M steps to the right of the boundary of F in at least one of the checker rows. If the block starting there is some $w_n \in \mathcal{W}_{\text{unif}}$ we also know from Rule (R3) that the word seen in the tester row has to be the label of a valid path passing through $v_{(n \mod |V_G|)+1}$ at this very coordinate. Therefore we may change the content of the tester row from this coordinate onward to the right end of the defect by filling in the label of a shortest valid path of length a multiple of M connecting $v_{(n \mod |V_G|)+1}$ to our special vertex v_1 followed by a concatenation of copies of the block w_0. The content of the two checker rows outside F is changed accordingly by filling in corresponding elements of $\mathcal{W}_{\text{unif}} \setminus \{w_0\}$ (followed by a concatenation of copies of w_0) always respecting Rule (R3). This modification takes place entirely outside the finite shape F and results in a shortened defect that ends at most $t + M$ steps to the right of F's boundary. A similar argument applies to defects that extend far to the left of the boundary of F (without ever re-entering F). In this case we change the content of the tester row immediately to the left of the starting coordinate of the first block from $\mathcal{W}_{\text{unif}}$ intersecting F in at least one of the two checker rows. We choose a shortest valid path of length a multiple of M connecting v_1 to the vertex coded by this block and put the path's label right-flush into the tester row. Note that again this path/word can be chosen to have length (at most) t. We adjust the blocks in both checker rows sitting above and below it to match the chosen path applying Rule (R3). Finally we fill any remaining space in the checker rows as well as in the tester row further to the left (until the left end of the original defect zone) with copies of w_0. This ensures that the shortened defect does not extend more than $t+M$ steps to the left of the boundary of F. Lastly, using the same technique, we address the case of a defect leaving F and carrying on for a (long) stretch of at least $2t+5M$ coordinates outside F before re-entering F: Starting from both sides, we break the original defect into two – both of which do not extend more than $t+M$ coordinates horizontally outside F – and we separate those by filling the remaining space – horizontally stretching across a multiple of M coordinates – between the two (new) shortened defects with copies of w_0. Doing this for all defects which intersect F and replacing all defects outside F by pure concatenations of copies of w_0 in all three of their rows we get a new valid point $\widetilde{x}^{(P)} \in X$ with $\widetilde{x}^{(P)}|_F = P$ but none of the defects extending more than $t+M$ coordinates horizontally outside F. In particular this means that $\widetilde{x}^{(P)}|_{\mathbb{Z}^2\setminus(F+[(-2,-t-3M),(2,t+3M)])}$ does not see any defect and thus is a subconfiguration of a valid point in $Y_{\text{fill}}^{\mathbb{Z}}$.

The claim now follows from the fact that Y_{fill} is mixing. Considering each row of the pattern $\widetilde{x}^{(P)}|_{F+[(-2,-t-3M),(2,t+3M)]}$ separately, allows us to extend such a finite "slice" by an element from $\mathcal{W}_{\text{fill}}$ to arrive at the aligned w_0-background forced by y_{fill}. To be a little more specific, we are using the same (slightly varied) argument in three distinct situations: For every $j \in \mathbb{Z}$ with $(F+[(-2,-t-3M),(2,t+3M)]) \cap (\mathbb{Z}\times\{j\}) \neq \emptyset$ choose the horizontal coordinate $i \in \mathbb{Z}$ outside but closest to the right end of the pattern $\widetilde{x}^{(P)}|_{(F+[(-2,-t-3M),(2,t+3M)])\cap(\mathbb{Z}\times\{j\})}$ at which a block w_0 ends. Immediately to its right put the right-infinite sequence $(w_0)^n\, w_0^{(m)}\, (w_0)^\infty$ where $n \in \mathbb{N}$ with $M' \leq n\cdot M < M'+M$ and $0 \leq m < M$ satisfies $i+(\widetilde{M}+m)+1 \in M\cdot\mathbb{Z}$. Similarly put the left-infinite sequence $(w_0)^\infty\, w_0^{(m')}\, (w_0)^n$ (with $0 \leq m' < M$ such that $i' - (\widetilde{M}+m') \in M\cdot\mathbb{Z}$) immediately to the left of the largest coordinate $i' \in \mathbb{Z}$ to the left of $\widetilde{x}^{(P)}|_{(F+[(-2,-t-3M),(2,t+3M)])\cap(\mathbb{Z}\times\{j\})}$ where a block w_0 starts. Recall from Lemma 6.3 that the word w_0 appears syndetically in any globally admissible pattern in Y_{fill}. Thus both the coordinates i and i' are at most a distance $\widetilde{M}$ from the set $(F+[(-2,-t-3M),(2,t+3M)])\cap(\mathbb{Z}\times\{j\})$ and so the infinite concatenations of copies of w_0 that are aligned according to the grid $M\cdot\mathbb{Z}\times\{j\}$ start within a horizontal distance of less than $3M'+t$ from $F\cap(\mathbb{Z}\times\{j\})$. The third situation is the case where we have (at least) two pieces of $(F+[(-2,-t-3M),(2,t+3M)])\cap(\mathbb{Z}\times\{j\})$ separated by a horizontal distance larger than $6M'+2t$. Starting from both ends we fill this gap with a finite block of the form $(w_0)^n\, w_0^{(m)}\, (w_0)^{n'}\, w_0^{(m')}\, (w_0)^n$ using $n \in \mathbb{N}$ with $M' \leq n\cdot M < M'+M$ and $0 \leq m, m' < M$ as before so that the middle piece $(w_0)^{n'}$ ($n' \in \mathbb{N}$) sits on the grid $M\cdot\mathbb{Z}\times\{j\}$. This way we modified all rows of $\widetilde{x}^{(P)}$ that intersect $F+[(-2,-t-3M),(2,t+3M)]$. If we fill all remaining rows, i.e. the ones that do not intersect, with the point $y_{\text{fill}} \in Y_{\text{fill}}$ we obtain our point $x \in \mathcal{A}^{\mathbb{Z}^2}$ such that $x|_F = x^{(P)}|_F = P$ while $x|_{\mathbb{Z}^2\setminus(F+[(-2,-3M'-t),(2,3M'+t)])} = w^*|_{\mathbb{Z}^2\setminus(F+[(-2,-3M'-t),(2,3M'+t)])}$ as claimed. (The reader may check that x satisfies all local rules (R1)-(R3) and thus $x \in X$.) Therefore we define $M^* := 3M'+t$ which yields a finite gap size $g = 7M'+2t$ and proves X strongly irreducible.

□

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Conditions 4.2.(1) and 4.2.(2) incorporating the most important ideas while not being too long or sounding too clumsy.)

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References

[1] N. Aubrun and M. Sablik, *Simulation of effective subshifts by two-dimensional shifts of finite type*, preprint 2010, to appear in Acta Applicandae Mathematicae.
[2] P. Balister, B. Bollobás and A. Quas, *Entropy along convex shapes, random tilings and shifts of finite type*, Illinois J. Math. **46** (2002), no. 3, 781–795.
[3] F.Blanchard and G. Hansel, *Systèmes codés*, Theoret. Comput. Sci. **44** (1986), no. 1, 17–49.
[4] M. Boyle, R. Pavlov and M. Schraudner, *Multidimensional sofic shifts without separation and their factors*, Trans. Amer. Math. Soc. **362** (2010), no. 9, 4617–4653.
[5] B. Durand, A. Romashchenko and A. Shen, *Fixed-point tile sets and their applications*, J. Computer and System Sciences **78** (2012), no. 3, 731–764.
[6] D. Fiebig and U.-R. Fiebig, *Invariants for subshifts via nested sequences of shifts of finite type*, Ergodic Theory Dynam. Systems **21** (2001), no. 6, 1731–1758.
[7] M. Hochman, *On the dynamics and recursive properties of multidimensional symbolic systems*, Invent. Math. **176** (2009), no. 1, 131–167.
[8] L. Hurd, *The application of formal language theory to the dynamical behavior of cellular automata*, Ph.D. Thesis, Princeton University (1988).
[9] B. Kitchens, *Symbolic dynamics. One-sided, two-sided and countable state Markov shifts*, Universitext, Springer-Verlag, Berlin (1998).
[10] W. Krieger, *On sofic systems. I.*, Israel J. Math. **48** (1984), no. 4, 305–330.
[11] W. Krieger, *On the uniqueness of the equilibrium state*, Math. Systems Theory **8** (1974/75), no. 2, 97–104.
[12] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge (1995).
[13] R. Pavlov and M. Schraudner, *Classification of sofic projective subdynamics of multidimensional shifts of finite type*, to appear in Trans. Amer. Math. Soc.
[14] A. Quas and M. Schraudner, *Minimal projective subdynamics of $\mathbb{Z}^d$ shifts of finite type*, work in progress.
[15] M. Sablik and M. Schraudner, *Speed of convergence of non-sofic projective subdynamics of multidimensional shifts of finite type*, work in progress.
[16] M. Schraudner, *Projectional entropy and the electrical wire shift*, Discrete Contin. Dyn. Syst. **26** (2010), no. 1, 333–346.

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