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On the algebraic properties of the automorphism groups of countable state Markov shifts

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Abstract. We study the algebraic properties of automorphism groups of two-sided, transitive, countable state Markov shifts together with the dynamics of those groups on the shiftspace itself as well as on periodic orbits and the 1-point-compactification of the shiftspace.

We present a complete solution to the cardinality-question of the automorphism group for locally compact and non locally compact, countable state Markov shifts, shed some light on its huge subgroup structure and prove the analogue of Ryan's theorem about the center of the automorphism group in the non-compact setting.

Moreover we characterize the 1-point-compactification of locally compact, countable state Markov shifts, whose automorphism groups are countable and show that these compact dynamical systems are conjugate to synchronised systems on doubly-transitive points.

Finally we prove the existence of a class of locally compact, countable state Markov shifts whose automorphism groups split into a direct sum of two groups; one being the infinite cyclic group generated by the shift map, the other being a countably infinite, centerless group, which contains all automorphisms that act on the orbit-complement of certain finite sets of symbols like the identity.

Keywords: countable state Markov shift, SFT, coded system, automorphism group, Ryan's theorem, 1-point-compactification, thinned-out-graph

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1. *Basic definitions and outline of the paper*

Let $\mathcal{A}$ be a countably infinite set endowed with the discrete topology. The product space $\mathcal{A}^{\mathbb{Z}}$ (with product topology), consisting of all bi-infinite sequences of symbols from the alphabet $\mathcal{A}$, is a non-compact, totally disconnected, perfect metric space.

The (left-)shift map $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}},\ \sigma((x_i)_{i\in\mathbb{Z}}) := (x_{i+1})_{i\in\mathbb{Z}}$ is a homeomorphism. It induces some dynamics on $\mathcal{A}^{\mathbb{Z}}$ and $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ is called the full shift on $\mathcal{A}$.

Every pair (X, σ) of some shift-invariant, closed subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$ endowed with the induced subspace topology joined with the restriction of the shift map $\sigma = \sigma|_X$ yields a subshift. There is a countable set of clopen cylinders ${}_n[a_0 \dots a_m] := \{(x_i)_{i\in\mathbb{Z}} \in X \mid \forall\, 0 \le i \le m :\ x_{n+i} = a_i\}$ $(n \in \mathbb{Z},\ m \in \mathbb{N}_0)$ generating the topology on X. Two subshifts (X_1, σ_1) and (X_2, σ_2) are (topologically) conjugate, if there is a homeomorphism $\gamma :\ X_1 \to X_2$ that commutes with the shift maps $(\sigma_2 \circ \gamma = \gamma \circ \sigma_1)$. Then (X_1, σ_1) and (X_2, σ_2) are merely two presentations of the same topological dynamical object and we denote by $\mathrm{Pres}(X)$ the set of all subshift presentations of (X, σ), i.e. the set of all subshifts conjugate to (X, σ).

Let $G = (V, E)$ be a directed graph with vertex set V, edge set E together with the maps $\mathfrak{i}, \mathfrak{t} :\ E \to V$, where $\mathfrak{i}(e)$ gives the initial and $\mathfrak{t}(e)$ the terminal vertex of an edge $e \in E$. A subshift (X, σ) is called countable state Markov shift, if its set of subshift presentations contains an edge shift (X_G, σ), with $X_G := \big\{(x_i)_{i\in\mathbb{Z}} \in E^{\mathbb{Z}} \mid \forall\, i \in \mathbb{Z} :\ \mathfrak{t}(x_i) = \mathfrak{i}(x_{i+1})\big\}$ the set of bi-infinite walks along the edges of a countably infinite directed graph G $(|E| = \aleph_0)$ and σ acting on X_G. If not stated explicitly, all graphs are directed, having a countably infinite set of edges. W.l.o.g. all graphs considered are assumed to be essential, i.e. the in-degree and out-degree at every vertex is strictly positive. (X_G, σ) is called a graph presentation of (X, σ) and $\mathrm{Graph}(X)$ denotes the set of all graph presentations of (X, σ).

For every point $x \in X$ in a subshift (X, σ) and $m \le n \in \mathbb{Z}$ let $x_{[m,n]}$, $x_{[m,\infty)}$ and $x_{(-\infty,n]}$ respectively denote the block $x_m\, x_{m+1} \dots x_{n-1}\, x_n$, a right-infinite or a left-infinite ray of x. In an edge shift $x_{[m,n]}$ corresponds to a finite path of length $n - m + 1$, whereas $x_{[m,\infty)}$ and $x_{(-\infty,n]}$ are equivalent to right-infinite and left-infinite walks.

We define the language $\mathcal{B}(X) := \dot{\bigcup}_{m\in\mathbb{N}} \mathcal{B}_m(X)$ of a subshift (X, σ) as the disjoint union (denoted by $\dot{\cup}$ in this paper) of all sets of blocks $\mathcal{B}_m(X) := \big\{x_{[0,m-1]} \bigm| x \in X\big\} \subseteq \mathcal{A}^m$ $(m \in \mathbb{N})$. $|w|$ denotes the length and w^n $(n \in \mathbb{N}_0 \,\dot{\cup}\, \{\infty\})$ the n-times concatenation of a block $w \in \mathcal{B}(X)$.

A subshift (X, σ) is called locally compact, if X is locally compact. For countable state Markov shifts this implies the compactness of every cylinder set. An edge shift (X_G, σ) is locally compact, iff every vertex in G has finite in-degree and out-degree (G is locally finite).

A subshift (X, σ) is called (topologically) transitive, if X is irreducible, i.e. for every pair $u, w \in \mathcal{B}(X)$ of blocks there is a block $v \in \mathcal{B}(X)$, such that $u\, v\, w \in \mathcal{B}(X)$. An edge shift (X_G, σ) is transitive, iff G is strongly connected.

Let $\mathrm{Orb}(X) := \{\mathrm{Orb}(x) \mid x \in X\}$ the set of σ-orbits $\mathrm{Orb}(x) := \{\sigma^n(x) \mid n \in \mathbb{Z}\}$. Using the backward-orbit $\mathrm{Orb}^-(x) := \{\sigma^{-n}(x) \mid n \in \mathbb{N}_0\}$ and the forward-orbit $\mathrm{Orb}^+(x) := \{\sigma^n(x) \mid n \in \mathbb{N}_0\}$ we define the set of doubly-transitive points $\mathrm{DT}(X) := \big\{x \in X \mid \mathrm{Orb}^-(x),\ \mathrm{Orb}^+(x) \text{ both are dense in } X\big\}$. For transitive subshifts this set is non-empty and dense. Let $x \in \mathrm{DT}(X)$ then every block $w \in \mathcal{B}(X)$ is contained infinitely often in $x_{(-\infty,0]}$ and $x_{[0,\infty)}$.

Finally we define the set of periodic points $\mathrm{Per}(X) := \bigcup_{n\in\mathbb{N}} \mathrm{Per}_n(X) = \dot{\bigcup}_{n\in\mathbb{N}} \mathrm{Per}_n^0(X)$ under the action of σ, where $\mathrm{Per}_n(X)$ denotes the set of points of period n and $\mathrm{Per}_n^0(X)$ the set of points of least period n. For transitive, countable state subshifts $\mathrm{Per}(X)$ is a countable dense subset in X.

For further notions and some background on subshifts we refer to the monographs on symbolic dynamics by D. Lind and B. Marcus [**LM**] and by B. Kitchens [**Kit**].

Now we recall the fundamental definition of this paper: Let (X, σ) be some subshift. A map $\varphi : X \to X$ is called an automorphism (of σ), if φ is a self-conjugacy, i.e. a shiftcommuting homeomorphism from X onto itself. Obviously the set of automorphisms forms a group $\mathrm{Aut}(\sigma)$ under composition, which is an invariant of topological conjugacy reflecting the inner structure and symmetries of the subshift.

For subshifts of finite type (SFTs) there is an extensive and profound theory dealing with automorphisms (see e.g. [**BK1**], [**BK2**], [**BLR**], [**FieU1**], [**FieU2**], [**Hed**], [**KR1**] and [**KRW1**]) and leading to very deep and strong results concerning the conjugacy problem [**KRW3**], the FOG-conjecture, asking whether the kernel of the dimension representation of $\mathrm{Aut}(\sigma)$ is generated by automorphisms of finite order (see [**KRW2**]) or the LIFT-hypothesis, examining which actions on subsets of periodic points can be extended to automorphisms of the whole subshift (see [**KR2**]). The automorphism group of any nontrivial SFT is a countably infinite, residually finite group (therefore it cannot contain any infinite simple or any nontrivial divisible subgroup) with center isomorphic to $\mathbb{Z}$. It is discrete with respect to the compact-open topology, does not contain any finitely generated subgroups with unsolvable word problem, but admits embeddings of a great variety of other groups (see the short review in Section 4).

The automorphism groups of coded systems have been studied in [**FF2**] with quite different results (they are much smaller and can be stipulated explicitely; their center can be isomorphic to a wide range of abstract groups). Moreover the automorphism groups of coded systems often split into direct sums $\mathrm{Aut}(\sigma) = \langle\sigma\rangle \oplus H$. And examples can be constructed for a great variety of abstract groups H. A similar result is yet established only for very few SFTs, e.g. full p-shifts with p prime, where H equals the set of inert automorphisms, i.e. the kernel of the dimension group representation of $\mathrm{Aut}(\sigma)$.

To our knowledge there are up to now no published results on automorphisms of countable state Markov shifts. Trying to fill part of this gap, the present paper contains results from the author's Ph.D. thesis [**Sch**]. Assuming that all Markov shifts considered in this paper are transitive, our main results are as follows:

In Section 2 we completely determine the cardinality of $\mathrm{Aut}(\sigma)$. For non-locally compact countable state Markov shifts this cardinality is always $2^{\aleph_0}$ (Theorem 2.2), whereas for locally compact, countable state Markov shifts it can be either $\aleph_0$ (like for any SFT) or again $2^{\aleph_0}$ (Theorem 2.4). Moreover we characterize the property of $\mathrm{Aut}(\sigma)$ being countably infinite via several equivalent conditions (Theorem 2.4, Lemma 2.6, Corollary 2.8, Theorem 3.4).

In Section 3 we study the topological 1-point-compactification of locally compact, countable state Markov shifts with $\mathrm{Aut}(\sigma)$ countable. Those compact dynamical systems need no longer be expansive, that is, in general they are not conjugate to any subshift. Instead the property $\mathrm{Aut}(\sigma)$ countable is equivalent to expansiveness being restricted to doubly-transitive points (Theorem 3.4). Furthermore Proposition 3.5 implies the existence of an almost invertible 1-block-factor-map from the compactification onto some synchronised system.

Section 4 contains some results on the subgroup structure of $\mathrm{Aut}(\sigma)$. Like in the SFT-case we can realize lots of abstract groups via marker constructions. The gradual relaxation of compactness (i.e. going from countable state Markov shifts with **(FMDP)** to general locally compact and finally to non locally compact ones) shows up in a decrease of algebraic restrictions and an increase of possible subgroups. This makes it very difficult to describe $\mathrm{Aut}(\sigma)$ as an abstract group. The fact that $\mathrm{Aut}(\sigma)$ is in a natural way a subgroup of the symmetric group $\mathcal{S}_{\mathbb{N}}$ on countably many elements (Proposition 2.1) instantly prohibits some abstract groups from being realized as subgroups of $\mathrm{Aut}(\sigma)$ (see Example 4.1). In Proposition 4.2, however, we exhibit a large class of non locally compact countable state Markov shifts with an automorphism group that itself contains an isomorphic copy of $\mathcal{S}_{\mathbb{N}}$ as well as copies of all the subgroups of $\mathcal{S}_{\mathbb{N}}$. The strong implications of this result on the algebraic and group-theoretic restrictions on $\mathrm{Aut}(\sigma)$ are summarized in Corollary 4.3. Proposition 4.4 presents a class of locally compact countable state Markov shifts containing at least a copy of the group $\mathcal{S}_{\mathbb{N},\mathfrak{f}}$ of finite permutations on a countably infinite set inside their automorphism groups. Finally Theorem 4.6 shows that $\mathrm{Aut}(\sigma)$ countable already implies the existence of a formal zeta function and this in turn implies residual finiteness of $\mathrm{Aut}(\sigma)$ exactly as in the SFT-setting.

Stimulated by the well-known result of J. Ryan [**Rya1**],[**Rya2**] on the center of $\mathrm{Aut}(\sigma)$ being isomorphic to $\mathbb{Z}$ for SFTs, we are able to re-prove this theorem for non-compact Markov shifts in Section 5. Therefore $\mathrm{Aut}(\sigma)$ is again highly non-abelian (in contrast to the coded-systems-case) and the periodic-orbit representation is faithful on ${}^{\mathrm{Aut}(\sigma)}/_{\langle\sigma\rangle}$ (Corollary 5.3 and Theorem 5.4).

The similarities between SFTs and countable state Markov shifts with $\mathrm{Aut}(\sigma)$ countable found in Sections 2 to 5 give rise to the question how to distinguish between their automorphism groups. The last two sections of this paper are devoted to a direct sum decomposition (Theorem 6.4) of $\mathrm{Aut}(\sigma)$ for a certain subclass of locally compact countable state Markov shifts with countably infinite automorphism group. Such a decomposition $\mathrm{Aut}(\sigma) = \langle\sigma\rangle \oplus H$ is well established for coded systems [**FF2**] as mentioned above. For general SFTs we do not know of any such decomposition except in some special cases like full-shifts on an alphabet of prime cardinality p, where the automorphism group splits due to the dimension group representation $\delta : \mathrm{Aut}(\sigma_p) \to \mathrm{Aut}(s_p) \cong \mathbb{Z}$ into the cyclic group generated by the shift map being the image of δ and the centerless, normal subgroup of inert automorphisms being the kernel of δ (see e.g. [**Wag**] or [**KRW4**]). Unfortunately even this partial result is rather non-constructive due to the fact that the set of inert automorphisms is not fully understood for arbitrary SFTs. Therefore our result on locally compact Markov shifts and further SFT-investigations in this direction may give rise to the desired difference between the countable automorphism groups of SFTs and countable state Markov shifts and thus solve this open question.

As all these results show, the automorphism groups of countable state Markov shifts and SFTs are quite similar but differ a lot from those of coded systems. The fundamental structure of $\mathrm{Aut}(\sigma)$ is thus mainly governed by the Markov property whereas compactness seems to have much less influence.

2. *The cardinality of* $\mathrm{Aut}(\sigma)$

Let $\mathcal{S}_{\mathbb{N}}$ be the set of all bijective mappings from $\mathbb{N}$ (or generally any countably infinite set) onto itself. We call $\mathcal{S}_{\mathbb{N}}$ the full permutation group (on a countable set). Its cardinality is $2^{\aleph_0}$. By $\mathcal{S}_{\mathbb{N},\mathfrak{f}}$ we denote the subgroup of finite permutations, i.e. the set of all bijective mappings from $\mathbb{N}$ onto itself that fix all but finitely many elements. The cardinality of $\mathcal{S}_{\mathbb{N},\mathfrak{f}}$ is $\aleph_0$.

We use the notation $G \le H$ to indicate that G is a homomorphic image of the algebraic object H, e.g. G is a subgroup if H belongs to the category of groups.

Proposition 2.1. *The automorphism group of every transitive, countable state Markov shift is isomorphic to a subgroup of* $\mathcal{S}_{\mathbb{N}}$ *and has cardinality at most* $2^{\aleph_0}$.

Proof. Since the Markov shift (X,σ) is transitive, the countable set of periodic points $\mathrm{Per}(X)$ is dense in X and every automorphism $\varphi \in \mathrm{Aut}(\sigma)$ is uniquely determined by its action on $\mathrm{Per}(X)$. Therefore $\mathrm{Aut}(\sigma) \le \mathcal{S}_{\mathrm{Per}(X)} \cong \mathcal{S}_{\mathbb{N}}$. □

Now we can state the cardinality-result for non locally compact Markov shifts:

Theorem 2.2. *Every transitive, non locally compact, countable state Markov shift has an automorphism group of cardinality* $2^{\aleph_0}$.

Proof. Let $G = (V,E)$ be a graph presentation for the non locally compact Markov shift (X,σ). W.l.o.g. we may assume that there is a vertex $v \in V$ with infinite out-degree (the symmetric situation of a vertex with infinite in-degree can be treated via time-reversal, i.e. carrying out the following construction for the transposed graph).

Let $\{e_j \mid j \in \mathbb{N}\} \subseteq E$ be the set of edges starting at v. For every $j \notin 3\mathbb{N}$ choose a shortest path p_j from $\mathfrak{t}(e_j)$ back to v (G is strongly connected); p_j is empty, if $\mathfrak{t}(e_j) = v$. This gives an infinite set of distinct loops $l_j := e_j\, p_j$ at the vertex v. Use the edges e_j ($j \in 3\mathbb{N}$) as markers to define maps $\phi_i : X \to X$ ($i \in \mathbb{N}$) that interchange the blocks $l_{3i-2}\, l_{3i-1}\, e_{3i}$ and $l_{3i-1}\, l_{3i-2}\, e_{3i}$ in every point $x \in X$ and take no further action.

By construction no path p_j ($j \notin 3\mathbb{N}$) can contain an edge e_i ($i \in \mathbb{N}$). This guarantees that no loop l_j ($j \notin 3\mathbb{N}$) contains any edge e_{3i} and no two loops can overlap partially. Therefore every ϕ_i is well-defined. ϕ_i is an involutorial sliding-block-code with coding length $2\,|l_{3i-2}\, l_{3i-1}|+1$. So we have constructed a countable set of distinct automorphisms $\{\phi_i \mid i \in \mathbb{N}\} \subseteq \mathrm{Aut}(\sigma)$.

Next consider infinite products of the maps ϕ_i and show that for every 0/1-sequence $(a_k)_{k\in\mathbb{N}}$ there is a well-defined automorphism $\varphi_{(a_k)} := \prod_{i\in\mathbb{N}} {\phi_i}^{a_i}$:

Distinct automorphisms ϕ_i act on disjoint blocks ending with the symbol e_{3i}. The ϕ_i commute with each other, so $\varphi_{(a_k)}$ is a well-defined order 2 bijection from X onto X and obviously $\varphi_{(a_k)}$ commutes with the shift map.

To show that $\varphi_{(a_k)}$ (and ${\varphi_{(a_k)}}^{-1} = \varphi_{(a_k)}$) is continuous, it suffices to show that the zero-coordinate of the image is prescribed by a finite block of the preimage:

Fix $x \in X$. The symbol x_0 is unchanged unless it is part of some block $l_{3i-2}\, l_{3i-1}\, e_{3i}$ or $l_{3i-1}\, l_{3i-2}\, e_{3i}$. Let n be the length of a shortest path from $\mathfrak{t}(x_0)$ to the vertex v. Whenever $x_{n+1} \notin \{e_i \mid i \in \mathbb{N}\}$ we have $\big(\varphi_{(a_k)}(x)\big)_0 = x_0$. If $x_{n+1} = e_j$ for some $j \in \mathbb{N}$ the only automorphism in the product that can act on the zero-coordinate is ϕ_i with $i := \lceil \frac{j}{3} \rceil$. We have $\big(\varphi_{(a_k)}(x)\big)_0 = \big({\phi_i}^{a_i}(x)\big)_0$. Since ϕ_i is sliding-block, $\big(\varphi_{(a_k)}(x)\big)_0$ is determined by the knowledge of a finite block of x. As $\varphi_{(a_k)}$ commutes with σ, this proves continuity.

Two distinct 0/1-sequences $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ define distinct automorphisms $\varphi_{(a_k)}, \varphi_{(b_k)}$. For $i \in \mathbb{N}$ such that $a_i \neq b_i$, the point $x := (l_{3i-2}\, l_{3i-1}\, e_{3i}\, p_{3i})^\infty \in X$ (p_{3i} a shortest path from $\mathfrak{t}(e_{3i})$ back to v) has different images under $\varphi_{(a_k)}$ and $\varphi_{(b_k)}$. Therefore we have constructed a subgroup $\big\{\varphi_{(a_k)} \mid (a_k)_{k\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}\big\} \leq \mathrm{Aut}(\sigma)$ of cardinality $2^{\aleph_0}$. □

We remark that though all ϕ_i in the proof of Theorem 2.2 are sliding-block-codes, i.e. uniformly continuous maps, the infinite products $\varphi_{(a_k)}$ need not have bounded coding length and are in general merely continuous. Continuity itself is only guaranteed by the careful choice of the ϕ_i acting on disjoint blocks.

To answer the cardinality-question for locally compact, countable state Markov shifts we need the notion of a double path in a directed graph G:

A pair of two distinct paths p, q of equal length ($|p| = |q|$), connecting a common initial with a common terminal vertex is called a double path and is denoted $[p;q]$. By definition we have $[p;q] = [q;p]$ and in slight abuse of notation $\mathfrak{i}(p) = \mathfrak{i}(q)$ and $\mathfrak{t}(p) = \mathfrak{t}(q)$.

A set of double paths is pairwise edge-disjoint, if no edge (from G) is part of more than one double path in this set.

Definition 2.3. *A strongly connected, directed graph has the property* **(FMDP)***, if it contains at most* **F***initely* **M***any pairwise edge-disjoint* **D***ouble* **P***aths.*

Theorem 2.4. *Let (X, σ) be a transitive, locally compact, countable state Markov shift.* $\mathrm{Aut}(\sigma)$ *has cardinality* $\aleph_0$*, iff any (every) graph presentation of (X, σ) has* **(FMDP)**. *Otherwise* $\mathrm{Aut}(\sigma)$ *has cardinality* $2^{\aleph_0}$.

The proof of Theorem 2.4 is given in three steps:

Lemma 2.5. *Let (X_G, σ) be any graph presentation of a transitive, locally compact, countable state Markov shift on some directed graph G containing infinitely many, pairwise edge-disjoint double paths. Then* $\mathrm{Aut}(\sigma)$ *has cardinality* $2^{\aleph_0}$.

Proof. Since X_G is irreducible and locally compact, G has to be strongly connected and locally finite. Let $P := \{[p_i; q_i] \mid i \in \mathbb{N}\}$ be an infinite set of pairwise edge-disjoint double paths in G. For every $[p_i; q_i]$ choose a marker edge e_i starting at

$\mathfrak{t}(p_i) = \mathfrak{t}(q_i)$ that is not contained in this double path. This is possible, since both paths p_i, q_i may be extended by the same finite set of edges already contained in $[p_i; q_i]$ until they end at a vertex, at which an edge not contained in $[p_i; q_i]$ starts. Take such an edge as marker and use the enlarged double path in place of $[p_i; q_i]$.

Inductively we construct an infinite subset $Q \subseteq P$ of double paths (with adjoint markers) such that all marker edges are distinct and no one does occur in any of the double paths in Q: Let $Q := \emptyset$. Choose $[p; q] \in P$; define $Q := Q \,\dot\cup\, \{[p; q]\}$. Due to the local finiteness of G there are at most finitely many elements in the set P whose markers are part of $[p; q]$. After removing this finite subset, the element $[p; q]$ itself as well as the double path (if there is one) containing the marker of $[p; q]$ from P, we are left with a still infinite set. Choosing one of the remaining double paths we iterate this procedure to build up an infinite subset Q as desired. For simplicity of notation renumber the elements in Q to get $Q = \{[p_i; q_i] \mid i \in \mathbb{N}\}$.

For every 0/1-sequence $(a_k)_{k\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ define a map $\varphi_{(a_k)} : X_G \to X_G$ that interchanges every block $p_i\, e_i$ and $q_i\, e_i$ in a point in X_G, iff $a_i = 1$. Caused by edge-disjointness of the double paths $[p_i; q_i] \in Q$ and the use of the distinct markers e_i, being edge-disjoint from all elements in Q, no partial overlaps are possible and $\varphi_{(a_k)}$ is well-defined. $\varphi_{(a_k)}$ commutes with σ by construction. Furthermore $\varphi_{(a_k)}(X_G) \subseteq X_G$ and ${\varphi_{(a_k)}}^2 = \mathrm{Id}_{X_G}$, that is $\varphi_{(a_k)} = {\varphi_{(a_k)}}^{-1}$ is bijective.

Continuity of $\varphi_{(a_k)}$ is shown as in the proof of Theorem 2.2. The zero-coordinate of $x \in X_G$ is unchanged unless x_0 is part of a by definition of Q uniquely determined double path $[p_j; q_j] \in Q$. Looking at the finite block $x_{[1-|p_j|,|p_j|]}$ one can decide about $\big(\varphi_{(a_k)}(x)\big)_0$: Suppose $x_{[m,m+|p_j|]} = p_j\, e_j$ for some $1 - |p_j| \le m \le 0$ and $a_j = 1$, then the zero-coordinate of the image has to be the $(1-m)$-th symbol of the block q_j. Analogously for $x_{[m,m+|p_j|]} = q_j\, e_j$. In all other cases $\big(\varphi_{(a_k)}(x)\big)_0 = x_0$. Therefore $\varphi_{(a_k)}$ is a shiftcommuting homeomorphism.

Obviously distinct sequences $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ give rise to distinct maps $\varphi_{(a_k)} \ne \varphi_{(b_k)}$, because for $i \in \mathbb{N}$ such that $a_i \ne b_i$ the images $\varphi_{(a_k)}(x)$ and $\varphi_{(b_k)}(x)$ of a point $x \in X_G$ with $x_{[0,|p_i|]} = p_i\, e_i$ differ. This shows the existence of a subgroup $\big\{\varphi_{(a_k)} \,\big|\, (a_k)_{k\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}\big\} \le \mathrm{Aut}(\sigma)$ of cardinality $2^{\aleph_0}$. □

For the next lemma we need the notion of the F-skeleton of a bi-infinite sequence: Let $F \subseteq \mathcal{A}$ be a subset of some alphabet $\mathcal{A}$. The F-skeleton of a point $x \in \mathcal{A}^{\mathbb{Z}}$ is the partial map $\kappa_x : \mathbb{Z} \to F$, $\kappa_x(i) := \begin{cases} x_i & \text{if } x_i \in F \\ \uparrow & \text{otherwise} \end{cases}$ ($\uparrow$ signals an undefined value of κ_x).

Lemma 2.6. *A locally finite, strongly connected graph $G = (V, E)$ has property* **(FMDP)**, *iff there is a finite set $F \subseteq E$ of edges, such that every doubly-transitive walk along the edges of G is uniquely determined by its F-skeleton.*

Proof. W.l.o.g. we may assume $|E| = \aleph_0$, since otherwise $F := E$ is a good choice to prove the statement.

"$\Longleftarrow$": Suppose G does not have **(FMDP)**, then for every finite set $F \subsetneq E$ there is a double path $[p; q]$ that does not contain an edge from F (in fact there are infinitely many). The path p occurs infinitely often in every doubly-transitive walk. Exchanging one such block p by the block q gives another doubly-transitive walk, that obviously has the same F-skeleton.

"$\Longrightarrow$": Assume that $P := \{[p_n; q_n] \mid 1 \leq n \leq N\}$ is a maximal, finite set of pairwise edge-disjoint double paths in G (having **(FMDP)**). Let $F \subsetneq E$ be the union of all edges that show up in elements of P. F is a finite set. Suppose there are two doubly-transitive walks $x, y \in \mathrm{DT}(X_G)$ with the same F-skeleton. There are coordinates $i \leq j \in \mathbb{Z}$ such that $x_{i-1} = y_{i-1}$, $x_{j+1} = y_{j+1} \in F$, $x_k, y_k \notin F$ for all $i \leq k \leq j$ and $x_{[i,j]} \neq y_{[i,j]}$. This implies the existence of a double path $[x_{[i,j]}; y_{[i,j]}]$ of length $j - i + 1$ connecting $\mathfrak{t}(x_{i-1}) = \mathfrak{t}(y_{i-1})$ with $\mathfrak{i}(x_{j+1}) = \mathfrak{i}(y_{j+1})$, which is edge-disjoint to all elements in P, contradicting the maximality of P. □

This equivalent reformulation of the property **(FMDP)** is enough to finish the proof of Theorem 2.4:

Lemma 2.7. *Let the transitive, locally compact, countable state Markov shift (X, σ) be presented on some directed graph $G = (V, E)$. Suppose there is a finite set $F \subsetneq E$ of edges such that every doubly-transitive point in X is uniquely determined by its F-skeleton, then $\mathrm{Aut}(\sigma)$ is countably infinite.*

Proof. Again G has to be a strongly connected, locally finite graph with $|E| = \aleph_0$. As all powers of σ are distinct automorphisms, $\mathrm{Aut}(\sigma)$ has at least cardinality $\aleph_0$. Since $\mathrm{DT}(X)$ forms a dense subset in X, every automorphism $\varphi \in \mathrm{Aut}(\sigma)$ is uniquely determined by its action on the doubly-transitive points. It suffices to show that there are at most countably many restrictions $\varphi|_{\mathrm{DT}(X)}$ possible.

Let $F \subsetneq E$ be a finite set as stated in the lemma. X is locally compact, so every zero-cylinder ${}_0[e]$ $(e \in E)$ is compact-open. The preimages $\varphi^{-1}({}_0[e])$ are also compact-open and therefore can be expressed as the union of a finite set of cylinders. Select such a cover with (minimal) cardinality $m_f \in \mathbb{N}$ for all $f \in F$:

$$\varphi^{-1}\big({}_0[f]\big) = \bigcup_{i=1}^{m_f} {}_{n_{f,i}}[b_{f,i}] \quad \text{with } b_{f,i} \in \mathcal{B}(X) \text{ and } n_{f,i} \in \mathbb{Z}$$

As φ commutes with σ one gets:

$$\varphi\big({}_{n_{f,i}+k}[b_{f,i}]\big) \subseteq {}_k[f] = \bigcup_{j=1}^{m_f} \varphi\big({}_{n_{f,j}+k}[b_{f,j}]\big) \quad \forall\, 1 \leq i \leq m_f,\ k \in \mathbb{Z}$$

Specifying these finite preimage cylindersets $\big\{{}_{n_{f,i}}[b_{f,i}] \mid 1 \leq i \leq m_f\big\}$ for all $f \in F$ is equivalent to prescribing the whole F-skeleton of the image of every point in X under φ. Since an automorphism maps $\mathrm{DT}(X)$ onto itself, this fixes $\varphi|_{\mathrm{DT}(X)}$.

Let M be the set of all mappings $\mu : F \to \{C \subsetneq \mathcal{C}(X) \mid C \text{ finite}\}$, $f \mapsto \big\{{}_{n_{f,i}}[b_{f,i}] \mid 1 \leq i \leq m_f\big\}$ where $\mathcal{C}(X)$ denotes the countable set of all cylinders of X. Obviously $\{C \subsetneq \mathcal{C}(X) \mid C \text{ finite}\}$ is countable and so is M. Now a mapping $\mu \in M$ induces at most one automorphism, so there is an injection from $\mathrm{Aut}(\sigma)$ into M, proving $\mathrm{Aut}(\sigma)$ countable. □

Since the automorphism group is an invariant of topological conjugacy, its cardinality is independent of the subshift presentation one has chosen. In particular this shows the conjugacy-invariance of the property **(FMDP)** as claimed in Theorem 2.4: Either every or no graph presentation of a given transitive, locally compact, countable state Markov shift has **(FMDP)**.

As a direct consequence of Lemmata 2.5 and 2.7 we get a result about the compact-open topology on $\mathrm{Aut}(\sigma)$. This topology is build up from subbasis sets of the form $S(C,U) := \{\varphi \in \mathrm{Aut}(\sigma) \mid \varphi(C) \subseteq U\}$, with $C \subseteq X$ compact and $U \subseteq X$ open.

For SFTs the compact-open topology on $\mathrm{Aut}(\sigma)$ is known to be discrete (see [**Kit**], Observation 3.1.2), whereas for countable state Markov shifts this need not be true:

Corollary 2.8. *Let (X,σ) be a transitive, locally compact, countable state Markov shift. The compact-open topology on* $\mathrm{Aut}(\sigma)$ *is discrete, iff* $\mathrm{Aut}(\sigma)$ *has cardinality* $\aleph_0$.

Proof. "$\Longleftarrow$": Using the notation of Lemma 2.7, every automorphism $\varphi \in \mathrm{Aut}(\sigma)$ is uniquely determined by fixing the finite sets of cylinders $\left\{{}_{n_{f,i}}[b_{f,i}] \;\middle|\; 1 \leq i \leq m_f\right\}$ for all $f \in F$. Since φ induces a bijection on the periodic points, it is not possible to have another automorphism, whose preimage cylindersets contain that of φ for all $f \in F$. Therefore the singleton $\{\varphi\}$ can be expressed as a finite intersection of subbasis sets:

$$\bigcap_{f\in F} S\Big(\bigcup_{i=1}^{m_f} {}_{n_{f,i}}[b_{f,i}], {}_0[f]\Big) = \bigcap_{f\in F} \Big\{\phi \in \mathrm{Aut}(\sigma) \;\Big|\; \phi\big(\bigcup_{i=1}^{m_f} {}_{n_{f,i}}[b_{f,i}]\big) \subseteq {}_0[f]\Big\} = \{\varphi\}$$

"$\Longrightarrow$": Suppose $\mathrm{Aut}(\sigma)$ is not countable. Then every graph presentation contains an infinite set of pairwise edge-disjoint double paths. Finite intersections of subbasis sets fix the action of an automorphism merely on a finite set of these double paths. To single out an automorphism $\varphi_{(a_k)}$ as defined in Lemma 2.5 one would need to fix the action on all double paths, but this is only possible via infinite intersections of subbasis sets. □

So the property **(FMDP)** not only governs the cardinality but also the topological structure of $\mathrm{Aut}(\sigma)$ for locally compact, countable state Markov shifts. In Section 4 we will see that **(FMDP)** in addition has a heavy impact on the subgroup structure of the automorphism group.

3. *The 1-point-compactification of locally compact Markov shifts with* $\mathrm{Aut}(\sigma)$ *countable*

As for any locally compact topological space one defines the 1-point-compactification (X_0, σ_0) of a transitive, locally compact, countable state Markov shift (X,σ) by adding a single point, which is fixed under σ_0: $X_0 := X \,\dot\cup\, \{\infty\}$ denotes the unique Alexandroff-compactification of X (see [**Dug**], p. 246) and the homeomorphism $\sigma_0 : X_0 \to X_0$ is the canonical extension of the shift map with $\sigma_0|_X := \sigma$ and $\sigma_0(\infty) := \infty$.

We recall the definition of Gurevich metric: As the cylinder sets form a countable base of the topology on X, the 1-point-compactification X_0 is metrizable. Moreover there is an (up to uniform equivalence) unique metric $d_0 : X_0 \times X_0 \to \mathbb{R}^+$ which is consistent with the topology induced on X_0 by compactification of the topological space X, i.e. the completion of X with respect to this metric is X_0. The restriction $d := d_0|_X$ is called Gurevich metric. If a locally compact, countable state Markov shift (X, σ) is given in some graph presentation on $G = (V, E)$ there is an explicit formula for the Gurevich metric (see e.g. [**FF1**], page 627):

$$\forall\, x, y \in X : \quad d(x, y) := \sum_{n \in \mathbb{Z}} 2^{-|n|} \, |h(x_n) - h(y_n)|$$

where $h : E \to \{m^{-1} \mid m \in \mathbb{N}\}$ denotes any injective mapping from the edge set into the unit-fractions. The freedom in the choice of h corresponds to d (or d_0) being unique up to uniform equivalence.

We remark that in general σ_0 is no longer expansive with respect to the Gurevich metric. As a consequence although X_0 is still a zero-dimensional topological space, the compactification (X_0, σ_0) is a compact-metric dynamical system that need not be (conjugate to) any subshift.

D. Fiebig ([**FieD**], lemma 4.1) has shown that σ_0 is expansive, i.e. (X_0, σ_0) is a subshift, if and only if any (every) graph presentation of (X, σ) on a locally finite, strongly connected graph $G = (V, E)$ contains a finite set $F \subsetneq E$ of edges such that:

(1) Every bi-infinite walk along the edges of G contains an edge from F.

(2) For any pair of edges $c, d \in E$ and $n \in \mathbb{N}$ there is at most one path $p := e_1\, e_2 \ldots e_n$ such that $\mathfrak{i}(e_1) = \mathfrak{t}(c)$, $\mathfrak{t}(e_n) = \mathfrak{i}(d)$ and $e_i \in E \setminus F$ for all $1 \le i \le n$.

(3) For every edge $e_0 \in E$ there is at most one right-infinite ray $r := e_0\, e_1\, e_2 \ldots$ with $e_i \in E \setminus F$ for all $i \ge 1$ and at most one left-infinite ray $l := \ldots e_{-2}\, e_{-1}\, e_0$ with $e_i \in E \setminus F$ for all $i \le -1$.

Lets have a look at property **(2)** first:

Lemma 3.1. *A strongly connected, locally finite graph G has property* **(2)***, iff it has* **(FMDP)**.

Proof. "$\Longrightarrow$": Suppose there are infinitely many pairwise edge-disjoint double paths in G. To fulfill **(2)** the set F has to contain at least one edge from every double path. This contradicts the finiteness of F.

"$\Longleftarrow$": Let P be a maximal finite set of pairwise edge-disjoint double paths in G. Define $F := \{e \in E \mid \exists\, [p; q] \in P : \ e \in p \ \vee \ e \in q\} \subsetneq E$ to be the union of all edges that occur in elements of P. Since P was maximal, every double path in G contains an edge from the finite set F, i.e. F satisfies property **(2)**. □

Putting together Theorem 2.4, Lemma 3.1 and the result by D. Fiebig ([**FieD**], lemma 4.1) we get:

COROLLARY 3.2. *The automorphism group of every transitive, locally compact, countable state Markov shift having an expansive 1-point-compactification is countably infinite.*

REMARK: There is another, more direct proof for this corollary, that does not refer to a graph presentation, but shows that even the set of endomorphisms $\mathrm{End}(\sigma)$ (continuous, shiftcommuting maps from X to itself) is countable:

Under the assumptions of Corollary 3.2 the 1-point-compactification (X_0, σ_0) is (conjugate to) a compact subshift. Due to the theorem of Curtis-Hedlund-Lyndon [**Hed**] every endomorphism $\phi_0 : X_0 \to X_0$ is a sliding-block-code. Therefore $\mathrm{End}(\sigma_0)$ is at most countable. In addition there is a canonical injection $\varepsilon : \mathrm{End}(\sigma) \to \mathrm{End}(\sigma_0)$, $\phi \mapsto \phi_0$ such that $\phi_0|_X = \phi$ and $\phi_0(\infty) = \infty$, proving $\mathrm{End}(\sigma)$ countable.

In the following lemma we show that the three properties in D. Fiebig's characterisation of expansiveness are not independent from each other. In fact **(3)** already implies **(1)**, forcing the equivalence between σ_0 being expansive and properties **(2)** and **(3)** alone.

LEMMA 3.3. *Every strongly connected, locally finite graph containing a finite set of edges that fulfill* **(3)***, automatically satisfies property* **(1)***.*

Proof. Let $G = (V, E)$ be a directed graph as desired and $F \subsetneq E$ be a finite set satisfying **(3)**; X_G denotes the set of bi-infinite walks along the edges of G.

Suppose property **(1)** could not be fulfilled, i.e. there is an infinite set $W := \{w^{(i)} \in X_G \mid i \in \mathbb{N}\}$ of bi-infinite walks, such that no finite set of edges is enough to mark all elements in W. W.l.o.g. assume that no two elements $w^{(i)}$, $w^{(j)} \in W$ $(i \neq j \in \mathbb{N})$ differ only by some translation $(\forall\, k \in \mathbb{Z} :\ \sigma^k(w^{(i)}) \neq w^{(j)})$ and no $w^{(i)}$ contains an edge from F.

To show that the elements of W are even pairwise edge-disjoint, suppose there is an edge $e \in w^{(i)}$ being also part of $w^{(j)}$ $(i \neq j \in \mathbb{N})$. Then $w^{(i)}$ and $w^{(j)}$ branch somewhere before (or after) e. This would give two distinct left-(right-)infinite walks ending (starting) at e that do not contain any edge from F. This clearly contradicts the assumption on F satisfying **(3)**, so the elements of W are pairwise edge-disjoint.

Let $I := \{\mathfrak{i}(f) \mid f \in F\} \subsetneq V$ be the finite set of initial vertices of all edges in F. For every $i \in \mathbb{N}$ choose an edge e_i in $w^{(i)}$ and a shortest path p_i connecting $\mathfrak{t}(e_i)$ with one of the vertices in I. By construction these paths do not contain any edge from F. Since I is finite and W is infinite, there is a vertex $v \in I$ at which two (in fact infinitely many) paths p_i, p_j end. Now the paths $e_i\, p_i$ and $e_j\, p_j$ are distinct $(e_i \neq e_j)$, end at the same vertex v and can be extended to left-infinite walks that do not contain any edge from F by attaching the left-infinite rays of $w^{(i)}$ and $w^{(j)}$ ending in $\mathfrak{i}(e_i)$, $\mathfrak{i}(e_j)$ respectively. Again this contradicts property **(3)**. □

Now we come to the main purpose of this section, which is to find a fundamental description of the graph-property **(FMDP)** in a priori conjugacy-invariant, purely dynamical terms. This finally results in a presentation-independent characterisation of locally compact, countable state Markov shifts with $\mathrm{Aut}(\sigma)$ countable. To achieve this we use the Gurevich metric.

We have seen that the 1-point-compactifications of locally compact, countable state Markov shifts with $\mathrm{Aut}(\sigma)$ countable need not be subshifts. σ_0 is expansive with respect to the Gurevich metric, iff in addition to property **(FMDP)** the Markov shift has also property **(3)**. The following theorem exposes what can be said about the 1-point-compactification in the absence of **(3)**:

THEOREM 3.4. *For transitive, locally compact, countable state Markov shifts* (X,σ) *property* **(FMDP)** *is equivalent to* σ_0 *being expansive (with respect to the Gurevich metric) on the doubly-transitive points, i.e. there is an expansivity constant* $c>0$ *such that the set of* c*-shadowing points* $T_c(x) := \{y\in X \mid \forall\, n\in\mathbb{Z}:\ d(\sigma^n(x),\sigma^n(y))\le c\}$ *is an one-element set for all* $x\in \mathrm{DT}(X)$. *In other words:* $\forall\, x\in\mathrm{DT}(X),\ y\in X:\ x\ne y \Rightarrow \exists\, n\in\mathbb{Z}:\ d(\sigma^n(x),\sigma^n(y))>c$.

Proof. "$\Longrightarrow$": Assume $G=(V,E)$ is a graph presentation for the Markov shift (X,σ) having **(FMDP)**. Following from Lemma 2.6, there is a finite set of edges $F\subsetneq E$ uniquely determining every doubly-transitive point in X via its F-skeleton. For a given injective mapping $h:\ E\to\{m^{-1}\mid m\in\mathbb{N}\}$ inducing the Gurevich metric, one defines $c:=\frac{1}{2}\min_{f\in F}\{\frac{1}{m}-\frac{1}{m+1}\ \big|\ m=h(f)^{-1}\}$. Since F is finite, $c>0$. For $x,y\in X$, $x_0\in F$ and $x_0\ne y_0$ we have the estimate:

$$d(x,y)\ge |h(x_0)-h(y_0)|\ge \frac{1}{m}-\frac{1}{m+1}\ge 2c>c \quad \text{with } m:=h(x_0)^{-1}$$

To obtain $d(\sigma^n(x),\sigma^n(y))\le c$ for all $n\in\mathbb{Z}$, the F-skeleton of x and y have to agree. So for $x\in\mathrm{DT}(X)$ this implies $x=y$ and therefore $T_c(x)=\{x\}$.

"$\Longleftarrow$": Now assume $G=(V,E)$ contains infinitely many pairwise edge-disjoint double paths. For every $c>0$ there exists a double path $[p;q]$ such that for all edges $e\in E$ contained in $[p;q]$ one has $h(e)\le\frac{c}{3}$ ($h:\ E\to\{m^{-1}\mid m\in\mathbb{N}\}$ as above). Every $x\in\mathrm{DT}(X)$ contains the block p infinitely often. Substituting any subset of these with q gives uncountably many distinct points $y\in X$. The following estimate shows that all of these shadow x in a distance $\le c$:

$$c\ge 3\max\{h(e)\ \big|\ e\in[p;q]\}\ge\sum_{j\in\mathbb{Z}}2^{-|j|}\max\{|h(x_i)-h(y_i)|\ \big|\ i\in\mathbb{Z}\}\ge$$

$$\ge\sum_{j\in\mathbb{Z}}2^{-|j|}\left|h\big((\sigma^n(x))_j\big)-h\big((\sigma^n(y))_j\big)\right| = d(\sigma^n(x),\sigma^n(y))\quad\forall\, n\in\mathbb{Z}$$

So $T_c(x)$ is uncountable and c cannot be an expansivity constant. □

Theorem 3.4 characterizes the dynamical systems that show up as 1-point-compactifications of transitive, locally compact, countable state Markov shifts (X,σ) with $\mathrm{Aut}(\sigma)$ countable as transitive, zero dimensional, compact-metric topological spaces equipped with a homeomorphism acting at least expansive on doubly-transitive points.

If (X, σ) additionally fulfills property **(3)**, every point is determined by its F-skeleton (for some $F \subsetneq E$ finite) and the homeomorphism is (fully) expansive with respect to the Gurevich metric. Another result by D. Fiebig ([**FieD**], lemma 4.5) shows that in this case the 1-point-compactification is already (conjugate to) a synchronised system with at most one point not containing a synchronising block, i.e. $\mathrm{SYN}(X_0) \supseteq X_0 \setminus \{\infty\}$.

Recall that a compact, transitive subshift (Y, σ_Y) is called synchronised system, if it has at least one synchronising block $w \in \mathcal{B}(Y)$ such that whenever $u\,w,\ w\,v \in \mathcal{B}(Y)$ are admissible, then $u\,w\,v \in \mathcal{B}(Y)$ is also admissible [**BH**]. $\mathrm{SYN}(Y) \subseteq Y$ denotes the set of points containing some synchronising block.

Property **(FMDP)** alone implies almost-conjugacy to a synchronised system:

PROPOSITION 3.5. *Let (X, σ) be a transitive, locally compact, countable state Markov shift with $\mathrm{Aut}(\sigma)$ countable. There is an almost-invertible 1-block-factor-code $\kappa : (X_0, \sigma_0) \to (Y, \sigma_Y)$ from the 1-point-compactification onto a synchronised system with $\mathrm{SYN}(Y) \supseteq Y \setminus \kappa(\infty)$, that is $\kappa|_{\mathrm{DT}(X_0)} : (\mathrm{DT}(X_0), \sigma_0|_{\mathrm{DT}(X_0)}) \to (\mathrm{DT}(Y), \sigma_Y|_{\mathrm{DT}(Y)})$ is a topological conjugacy on the doubly-transitive points.*

Proof. Let $G = (V, E)$ be a graph presentation for (X, σ) and $F \subsetneq E$ a finite set of edges determining every doubly-transitive point via its F-skeleton. Define $A := F \,\dot{\cup}\, \{\uparrow\}$. The skeleton map $\kappa :\ X_0 \to A^{\mathbb{Z}} :\ \big(\kappa(x)\big)_i := \begin{cases} x_i & \text{if } x_i \in F \\ \uparrow & \text{otherwise} \end{cases}$ $\forall\, x \in X,\ i \in \mathbb{Z}$ and $\kappa(\infty) := \uparrow^{\infty}$ is a 1-block-map, thus continuous and shiftcommuting. As X_0 is compact, so is $Y := \kappa(X_0)$; $(Y, \sigma_Y) \subseteq (A^{\mathbb{Z}}, \sigma)$ is a compact subshift.

Every symbol $f \in F$ is a synchronising block in (Y, σ_Y): Let $\tilde{x}, \tilde{y} \in Y$ with $\tilde{x}_0 = f = \tilde{y}_0$. Since X is given in graph presentation, all preimages $x \in \kappa^{-1}(\tilde{x}) \subseteq X$, $y \in \kappa^{-1}(\tilde{y}) \subseteq X$ can be merged at their common zero-coordinate f to form a new point $z \in X$ with $z_{(-\infty,0]} = x_{(-\infty,0]}$ and $z_{[0,\infty)} = y_{[0,\infty)}$. By definition of κ one gets $\tilde{z} := \kappa(z)$ with $\tilde{z}_{(-\infty,0]} = \tilde{x}_{(-\infty,0]}$ and $\tilde{z}_{[0,\infty)} = \tilde{y}_{[0,\infty)}$, so f is in fact synchronizing for Y and every point in $Y \setminus \{\uparrow^{\infty}\}$ sees a synchronising symbol.

It remains to show that $\kappa|_{\mathrm{DT}(X_0)}$ is a topological conjugacy: As $X_0 = X \,\dot{\cup}\, \{\infty\}$ we have $\mathrm{DT}(X_0) = \mathrm{DT}(X)$. Every point $y \in \mathrm{DT}(Y)$ contains infinitely many edges from F in its left-infinite and its right-infinite ray. The blocks $\uparrow^n$ $(n \in \mathbb{N})$ between those edges can be decoded uniquely to paths in G. There is an unique preimage $x \in \mathrm{DT}(X)$ with $\kappa(x) = y$. This proves bijectivity of $\kappa|_{\mathrm{DT}(X_0)}$.

Finally the inverse map $(\kappa|_{\mathrm{DT}(X_0)})^{-1}$ is continuous with respect to the induced topologies on $\mathrm{DT}(X_0) \subseteq X$ and $\mathrm{DT}(Y) \subseteq Y$: Let $y \in \mathrm{DT}(Y)$ and $W(x) \subseteq \mathrm{DT}(X)$ some neighbourhood of $x := \kappa^{-1}(y) \in X$. For $m, n \in \mathbb{N}$ large enough, $W(x)$ contains a cylinder ${}_{-n}[x_{-n} \ldots x_0 \ldots x_m] \cap \mathrm{DT}(X)$ with $x_{-n},\ x_m \in F$. Its image $V(y) := \kappa\big({}_{-n}[x_{-n} \ldots x_0 \ldots x_m] \cap \mathrm{DT}(X)\big)$ is compact-open, contains y and satisfies $\kappa^{-1}\big(V(y)\big) = {}_{-n}[x_{-n} \ldots x_0 \ldots x_m] \cap \mathrm{DT}(X) \subseteq W\big(\kappa^{-1}(y)\big) \subseteq \mathrm{DT}(X)$. □

4. *On the subgroup structure of* $\mathrm{Aut}(\sigma)$

Using marker constructions lots of abstract groups have been embedded into the automorphism groups of SFTs (see [**Hed**], [**BLR**], [**KR1**], [**Kit**]) to show the rich and diverse structure of $\mathrm{Aut}(\sigma)$ and to exhibit some algebraic restrictions. In this context we call an abstract group H a subgroup of $\mathrm{Aut}(\sigma)$, if $\mathrm{Aut}(\sigma)$ contains a subgroup isomorphic to H. Since it is possible to carry over the whole concept of marker automorphisms to countable state Markov shifts, all subgroups realized in this way inside the automorphism groups of SFTs also show up in the non-compact setting.

Therefore we already have the following results on possible subgroups of $\mathrm{Aut}(\sigma)$: According to [**BLR**] the automorphism group of every transitive (mixing), countable state Markov shift contains any direct sum of countably many finite groups, the direct sum of countably many copies of $\mathbb{Z}$, the free group on countably many generators and any free product of finitely many cyclic groups, as well as all of their subgroups. Moreover the fundamental group of any 2-manifold and any countable, locally finite, residually finite group is a subgroup in $\mathrm{Aut}(\sigma)$ ([**KR1**]). To say it in few words, this review of results tells us that the subgroup structure of $\mathrm{Aut}(\sigma)$ for countable state Markov shifts is at least as rich as that found for SFTs.

But what about groups realizable only in the countable state case and what about remaining, relaxed and new restrictions and algebraic properties of $\mathrm{Aut}(\sigma)$?

Example 4.1. *Some groups not embeddable into* $\mathrm{Aut}(\sigma)$

Recall from Proposition 2.1 that – even in the non locally compact setting – the automorphism group of any transitive, countable state Markov shift is a subgroup of $\mathcal{S}_{\mathbb{N}}$. Using the work of N.G. de Bruijn as well as that of M. Kneser and S. Swierczkowski one can exclude certain abstract groups from being subgroups of $\mathrm{Aut}(\sigma)$: The group of all finite permutations on a set of cardinality $2^{\aleph_0}$ ([**Bru1**], theorem 5.1) and the group $H := {}^{F}/_{F''}$, where F is a non-abelian free group with more than $2^{\aleph_0}$ generators, F' its commutator subgroup and F'' the commutator group of F' ([**KS**], theorem 2), cannot be realized in $\mathcal{S}_{\mathbb{N}}$ and thus are never contained inside the automorphism group of any transitive, countable state Markov shift.

On the contrary there is at least a class of non locally compact, countable state Markov shifts with $\mathcal{S}_{\mathbb{N}}$ itself occuring as a subgroup in $\mathrm{Aut}(\sigma)$. The automorphism groups of this class are hence universal in the sense that they contain a copy of the automorphism group of any transitive, countable state subshift with periodic points dense (apply the argument in the proof of Proposition 2.1). Prototype for this class is the full-shift $\mathcal{A}^{\mathbb{Z}}$ with $|\mathcal{A}| = \aleph_0$.

Proposition 4.2. *If a transitive, non locally compact, countable state Markov shift is presentable on a graph containing an infinite number of paths of fixed length connecting a common initial with a common terminal vertex, then* $\mathcal{S}_{\mathbb{N}}$ *is (isomorphic to) a subgroup of its automorphism group.*

Proof. Let $G = (V, E)$ be a graph presentation as assumed in the proposition, $k \in \mathbb{N}$ the smallest natural number such that there are two vertices $u, v \in V$ $(u = v$

allowed) with infinitely many distinct paths p_i ($i \in \mathbb{N}_0$) of length k between them. W.l.o.g. assume all paths p_i pairwise edge-disjoint. This is possible due to the fact that as k is chosen minimal any edge in E can only be part of a finite number of paths p_i (otherwise one could choose an infinite subset of shortened paths).

As G is strongly connected, there is a shortest path q connecting $v = \mathfrak{t}(p_i)$ with $u = \mathfrak{i}(p_i)$. Let $f \in E$ be the initial edge of p_0. For every permutation $\pi \in \mathcal{S}_{\mathbb{N}}$ define a map $\varphi_\pi : X_G \to X_G$ which scans a point and replaces every block $p_i\, q\, f$ with $p_{\pi(i)}\, q\, f$ ($i \in \mathbb{N}$). These are well-defined (f cannot occur in q or any p_i), bijective sliding-block-codes with memory and anticipation $\leq k + |q|$. Since $\varphi_\pi \circ \varphi_\tau = \varphi_{\pi\circ\tau}$ and ${\varphi_\pi}^{-1} = \varphi_{\pi^{-1}}$, we have constructed a set of automorphisms $\{\varphi_\pi \mid \pi \in \mathcal{S}_{\mathbb{N}}\} \leq \mathrm{Aut}(\sigma)$ isomorphic to $\mathcal{S}_{\mathbb{N}}$. □

We collect the strong implications of Proposition 4.2 in the following corollary. To say it in short: Most of the algebraic restrictions (e.g. residual finiteness, divisibility of subgroups, non-existence of finitely generated subgroups with unsolvable word problem; see [**BLR**], section 3) known for the automorphism groups of SFTs vanish completely for the special class of Markov shifts described above and therefore a lot of subgroups which are forbidden for SFTs show up.

Recall that the automorphism group is residually finite, if for every element $\varphi \in \mathrm{Aut}(\sigma)$, $\varphi \neq \mathrm{Id}_X$ there is a finite group H and a homomorphism $\alpha : \mathrm{Aut}(\sigma) \to H$ with $\alpha(\varphi) \neq 1_H$. This property excludes the existence of both infinite simple and nontrivial divisible subgroups.

An abstract group H is called simple, if it contains no normal subgroups other than the trivial group $\{1_H\}$ and H itself. An abstract group H is divisible, if for every element $h \in H$ and every $n \in \mathbb{N}$ there is an element $g \in H$ with $g^n = h$.

Corollary 4.3. *Let (X, σ) be a transitive, non locally compact, countable state Markov shift as in Proposition 4.2.*
Its automorphism group contains infinite simple groups and is thus not residually finite. Every countable group can be realized in $\mathrm{Aut}(\sigma)$. *In particular the divisible groups $\mathbb{Q}$ and $\mathbb{Z}(p^\infty) = \mathbb{Z}[1/p]/\mathbb{Z}$ (p prime) can be embedded. The automorphism group does contain finitely generated groups with unsolvable word problem. Moreover every abelian group of cardinality $2^{\aleph_0}$ (especially $\mathbb{R}$) occurs in* $\mathrm{Aut}(\sigma)$. *Finally its set of subgroups is closed under taking free products of any $2^{\aleph_0}$ of its elements.*

Proof. The existence of the infinite simple subgroup $\mathcal{A}_{\mathbb{N},\mathfrak{f}} \leq \mathcal{S}_{\mathbb{N}}$ (alternating group on a countably infinite set) within $\mathrm{Aut}(\sigma)$ prohibits residual finiteness: Let A be a subgroup of $\mathrm{Aut}(\sigma)$ which is isomorphic to $\mathcal{A}_{\mathbb{N},\mathfrak{f}}$ and let $\varphi \in A \setminus \{\mathrm{Id}_X\}$. Assume $\mathrm{Aut}(\sigma)$ residually finite, then there exists a group-homomorphism $\alpha : \mathrm{Aut}(\sigma) \to H$ with H finite and $\alpha(\varphi) \neq 1_H$. The restriction $\alpha|_A$ would thus be a nontrivial homomorphism of A into the finite group H. Its kernel has to be a normal subgroup of the simple group A and therefore the image $\alpha|_A(A)$ would be either A which is infinite or trivial which contradicts $\alpha(\varphi) \neq 1_H$.

Every group H operates on itself by (left-)translation $\alpha_g : H \to H,\ h \mapsto g\,h$

$\forall g \in H$. This yields a representation of H as a group of permutations on H. So $H \leq \mathcal{S}_H \cong \mathcal{S}_{\mathbb{N}} \leq \mathrm{Aut}(\sigma)$.

According to R. Lyndon and P. Schupp ([**LS**], theorem IV.7.2) there are finitely generated, countable groups with unsolvable word problem, e.g. $H := \langle a, b, c, d \mid a^{-i}\, b\, a^{i} = c^{-i}\, d\, c^{i} \text{ iff } i \in S \rangle$ where $S \subsetneq \mathbb{N}$ is a recursively enumerable, non recursive subset.

The last two statements follow from the work of N.G. de Bruijn: Let M be an infinite set of cardinality m then every abelian group of cardinality 2^m is embeddable into $\mathcal{S}_M$ ([**Bru2**], theorem 4.3) and the free product of 2^m copies of $\mathcal{S}_M$ can be embedded into $\mathcal{S}_M$ ([**Bru1**], theorem 4.2). □

Next we specify a larger class of transitive, (non) locally compact, countable state Markov shifts admitting at least an embedding of the restricted permutation group $\mathcal{S}_{\mathbb{N},\mathfrak{f}}$ into $\mathrm{Aut}(\sigma)$. For this we need a graph presentation containing a strongly connected, infinite, tree-like subgraph consisting of an infinite number of loops l_i $(i \in \mathbb{N})$ – the nodes of the tree – of uniform length and of paths $p_{i,j}$ and $p_{j,i}$ – the links between the nodes – connecting the loops l_i and l_j as indicated exemplary in the figure below.

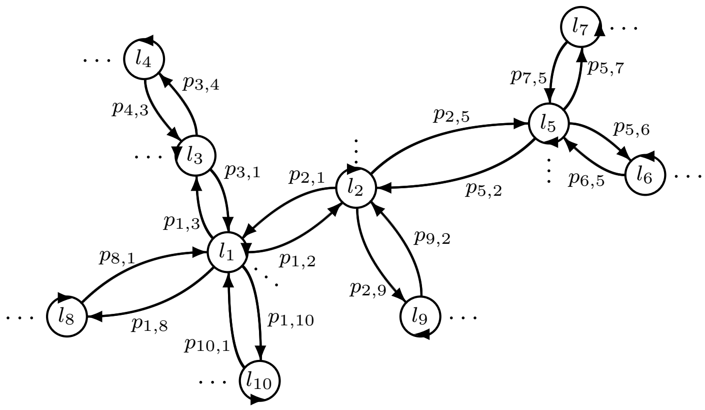


Figure 1. An infinite, strongly connected tree-like subgraph consisting of infinitely many loops l_i of some common length and paths $p_{i,j}$ and $p_{j,i}$ of globally bounded length connecting these loops.

If the length of all paths $p_{i,j}$ and $p_{j,i}$ in this subgraph is bounded globally, we can construct a subgroup of $\mathrm{Aut}(\sigma)$ isomorphic to $\mathcal{S}_{\mathbb{N},\mathfrak{f}}$. Obviously this class of Markov shifts contains the previously considered family of non locally compact Markov shifts (Proposition 4.2) as well as a subclass of locally compact, countable state Markov shifts. Prototypes for this class are (topological) random walks on $\mathbb{N}$ or on $\mathbb{Z}$ with steps $0, \pm 1$.

Proposition 4.4. *If some graph presentation of a transitive, countable state Markov shift contains an infinite set of loops* $L = \{l_i \mid i \in \mathbb{N}\}$ *of equal length, such* *that for every loop l_i there is, within a bounded distance, another loop l_j ($i > j \in \mathbb{N}$), i.e. there is a path $p_{i,j}$ connecting a vertex of l_i with one of l_j, a path $p_{j,i}$ connecting l_j back to l_i and both paths have length bounded by a global constant, then $\mathcal{S}_{\mathbb{N},\mathfrak{f}}$ can be embedded into the automorphism group.*

Proof. We distinguish between two cases: Either the graph presentation $G = (V, E)$, as assumed in the proposition, contains a loop $l \in L$ and an infinite subset $L' \subseteq L$ of loops having distance to l bounded by some constant $M \in \mathbb{N}$. Then G cannot be locally finite. G already fulfills the assumptions of Proposition 4.2, because using the elements in L' there are infinitely many distinct paths of length $\leq 3(|l|-1)+2M$ from one vertex in l back to this vertex. Thus not only $\mathcal{S}_{\mathbb{N},\mathfrak{f}}$ but even $\mathcal{S}_{\mathbb{N}}$ can be embedded into $\mathrm{Aut}(\sigma)$.

In the remaining case the tree-like subgraph consisting of the loops $l_i \in L$ and the paths $p_{i,j}$ and $p_{j,i}$ is locally finite. After choosing an appropriate infinite subset of L and renumbering its elements we can find an infinite chain of pairwise vertex-disjoint loops l_k ($k \in \mathbb{N}$) being connected via paths $p_{k,k+1}$ and $p_{k+1,k}$ of length bounded by $M \in \mathbb{N}$ such that every path $p_{k,k+1}$ and $p_{k+1,k}$ is vertex-disjoint from any l_i with $i \notin \{k, k+1\}$. W.l.o.g. choose $p_{k,k+1}$ and $p_{k+1,k}$ of minimal length, such that $\mathfrak{i}(p_{1,2}) = \mathfrak{t}(p_{2,1}) = \mathfrak{i}(l_1)$ and $\mathfrak{t}(p_{k,k+1}) = \mathfrak{i}(p_{k+1,k}) = \mathfrak{i}(p_{k+1,k+2}) = \mathfrak{t}(p_{k+2,k+1}) = \mathfrak{i}(l_{k+1})$ for all $k \in \mathbb{N}$. For the rest of the proof it suffices to look at such a linear chain.

Let $N := |l_k|$ be the common length of all loops l_k. Define a countably infinite set of closed paths

$$b_k := p_{k,k+1}\, p_{k+1,k}\, l_k\, (p_{k,k+1}\, p_{k+1,k})^{\frac{(2M)!}{|p_{k,k+1}p_{k+1,k}|}-1} \qquad (k \in \mathbb{N})$$

of uniform length $(2M)!+N$ which, due to the minimality of $p_{k,k+1}$ and $p_{k+1,k}$, allow no nontrivial overlaps: As b_k contains no vertex $\mathfrak{i}(l_i)$ with $i \notin \{k, k+1\}$ any overlap between blocks b_k and b_j with $|k-j| > 1$ is excluded. Moreover considering the occurences of the vertices $\mathfrak{i}(l_k)$ and $\mathfrak{i}(l_{k+1})$ inside b_k prohibits any overlap between blocks b_k and b_{k+1} and the only overlap possible between two blocks b_k consists of the prefix respectively suffix $p_{k,k+1}\, p_{k+1,k}$.

Furthermore cyclically shifting the blocks b_k by $|p_{k,k+1}|$ symbols to the left yields

$$\tilde{b}_k := p_{k+1,k}\, l_k\, (p_{k,k+1}\, p_{k+1,k})^{\frac{(2M)!}{|p_{k,k+1}p_{k+1,k}|}-1}\, p_{k,k+1} \qquad (k \in \mathbb{N})$$

Obviously $\mathfrak{i}(\tilde{b}_k) = \mathfrak{t}(\tilde{b}_k) = \mathfrak{i}(b_{k+1}) = \mathfrak{t}(b_{k+1})$ and $|\tilde{b}_k| = |b_k| = (2M)! + N$ for all $k \in \mathbb{N}$.

For every $k \in \mathbb{N}$ define a $\big((2M)! + N - 1, (2M)! + N - 1\big)$-sliding-block-code $\phi_{(k,k+1)} : X \to X$, which scans a point and replaces every block $\tilde{b}_k$ by b_{k+1} as well as every block b_{k+1} by $\tilde{b}_k$. $\phi_{(k,k+1)}$ is well-defined, since occurences of $\tilde{b}_k$ and b_{k+1} cannot overlap (again consider the occurences of vertices $\mathfrak{i}(l_i)$ $i \in \mathbb{N}$ inside $\tilde{b}_k$ and b_{k+1}; recall that neither $p_{k,k+1}$ nor $p_{k+1,k}$ contains $\mathfrak{i}(l_{k+2})$ and neither $p_{k+1,k+2}$ nor $p_{k+2,k+1}$ contains $\mathfrak{i}(l_k)$). By definition the maps $\phi_{(k,k+1)}$ are continuous, shiftcommuting involutions, hence automorphisms. Moreover $\phi_{(k,k+1)}\big((\tilde{b}_k)^\infty\big) = (b_{k+1})^\infty$ and $\phi_{(k,k+1)}\big((b_{k+1})^\infty\big) = (\tilde{b}_k)^\infty$ imply

$\phi_{(k,k+1)}\big(\mathrm{Orb}\big((b_k)^\infty\big)\big) = \mathrm{Orb}\big((b_{k+1})^\infty\big)$, $\phi_{(k,k+1)}\big(\mathrm{Orb}\big((b_{k+1})^\infty\big)\big) = \mathrm{Orb}\big((b_k)^\infty\big)$ and $\phi_{(k,k+1)}\big(\mathrm{Orb}\big((b_i)^\infty\big)\big) = \mathrm{Orb}\big((b_i)^\infty\big)$ for all $i \notin \{k, k+1\}$. The family of automorphisms $\big(\phi_{(k,k+1)}\big)_{k\in\mathbb{N}}$ acts on $\mathcal{O} := \big\{\mathrm{Orb}\big((b_k)^\infty\big) \mid k \in \mathbb{N}\big\}$ like the set of transpositions $\big((k, k+1)\big)_{k\in\mathbb{N}}$ does on $\mathbb{N}$. One easily checks that different presentations of a finite permutation on $\mathcal{O}$ as finite products of the $\phi_{(k,k+1)}$ yield the same automorphism. As any permutation in $\mathcal{S}_{\mathbb{N},\mathfrak{f}} \cong \langle (k, k+1) \mid k \in \mathbb{N}\rangle$ is presentable as a finite product of transpositions, the set $\big\{\phi_{(k,k+1)} \mid k \in \mathbb{N}\big\}$ generates a subgroup of $\mathrm{Aut}(\sigma)$ isomorphic to $\mathcal{S}_{\mathbb{N},\mathfrak{f}}$. □

Corollary 4.5. *The automorphism groups of topological Markov shifts satisfying the assumptions of Proposition 4.4 contain infinite simple subgroups and are thus not residually finite.*

Proof. The alternating group $\mathcal{A}_{\mathbb{N},\mathfrak{f}}$ on a countably infinite set is an infinite simple subgroup of $\mathcal{S}_{\mathbb{N},\mathfrak{f}}$. Hence $\mathrm{Aut}(\sigma) \geq \mathcal{A}_{\mathbb{N},\mathfrak{f}}$ is not residually finite. □

Finally we show that property **(FMDP)** (or equivalently $\mathrm{Aut}(\sigma)$ countably infinite) implies all of the restrictions on the subgroup structure of $\mathrm{Aut}(\sigma)$ known for SFTs:

Theorem 4.6. *For every transitive, locally compact, countable state Markov shift property* **(FMDP)** *forces the existence of a formal zetafunction, i.e. for any given period there are at most finitely many periodic points.*

Proof. Let $G = (V, E)$ be a strongly connected, locally finite graph presenting the Markov shift and let $F \subsetneq E$ be a finite set of edges such that every double path in G (having **(FMDP)**) contains an element from F. Suppose X_G has no formal zetafunction.

There is a smallest period length $k \in \mathbb{N}$ with $|\mathrm{Per}_k(X_G)| = \aleph_0$ and G has infinitely many simple loops of length k. (A path/loop is called simple, if it has no proper closed subpath.) As G is locally finite, one can choose an infinite set $L := \{l_i \mid i \in \mathbb{N}_0\}$ of these simple loops that are pairwise vertex-disjoint and in addition edge-disjoint from the set F (local finiteness of G implies that any edge can be a part of at most finitely many loops of fixed length). For every $i \in \mathbb{N}$ choose a shortest path p_i from $\mathfrak{i}(l_0)$ to $\mathfrak{i}(l_i)$ and a shortest path q_i from $\mathfrak{i}(l_i)$ back to $\mathfrak{i}(l_0)$. Since $l_0\, p_i$ and $p_i\, l_i$ form a double path, p_i has to contain an edge from F. The same is true for q_i. Using a pigeon hole argument, one gets a pair of subsets M_1, $M_2 \subseteq F$ such that there exists an infinite subset:

$$L' := \{l_i \in L \mid i \in \mathbb{N} \wedge (\forall f:\ f \in M_1 \Leftrightarrow f \in p_i) \wedge (\forall f:\ f \in M_2 \Leftrightarrow f \in q_i)\}$$

For notational simplicity renumber the elements in $L' = \{l_i \mid i \in \mathbb{N}\}$ (as well as their paths p_i, q_i) consecutively.

By construction the elements in M_1 occur exactly once in all paths p_i. Moreover the order of these occurences in every p_i is independent of $i \in \mathbb{N}$. Analogously for M_2 and q_i. Look at the shortened paths $\tilde{p}_i$ being the suffix of p_i, connecting the terminal vertex of the last edge from M_1 with $\mathfrak{i}(l_i)$ and $\tilde{q}_i$ being the prefix of q_i connecting $\mathfrak{i}(l_i)$ to the initial vertex of the first edge from M_2. Obviously no $\tilde{p}_i$, $\tilde{q}_i$ does contain an edge from F, but all of them start (end) at a common vertex. Another pigeon hole argument gives two distinct indices $i \neq j \in \mathbb{N}$ such that $|\tilde{p}_i| + |\tilde{q}_i| = |\tilde{p}_j| + |\tilde{q}_j| + m \cdot k$ with $m \in \mathbb{N}_0$. The double path $[\tilde{p}_i\, \tilde{q}_i; \tilde{p}_j\, {l_j}^m\, \tilde{q}_j]$ contradicts the assumption on F. Therefore X_G has a formal zetafunction. □

Theorem 4.6 allows us to get most of the restrictive results on the algebraic structure of the automorphism groups of SFTs from section 3 in [**BLR**] by simply copying the proofs using only the existence of a zetafunction. We included the proof just for completeness:

COROLLARY 4.7. *Let (X, σ) be a transitive, locally compact, countable state Markov shift with* $\mathrm{Aut}(\sigma)$ *countable. Then the automorphism group is residually finite. Thus* $\mathrm{Aut}(\sigma)$ *neither contains any nontrivial divisible nor any infinite simple subgroup. This excludes some abstract countable (abelian) groups, like* $\mathcal{A}_{\mathbb{N},\mathfrak{f}}$, $\mathrm{PSL}_n(\mathbb{Q})$ *(the projective unimodular groups over the rationals for $2 \leq n \in \mathbb{N}$),* $\mathbb{Q}$, $\mathbb{Z}(p^\infty)$ *(p prime). A subgroup of* ${}^{\mathbb{Q}}/_{\mathbb{Z}}$ *is realized in* $\mathrm{Aut}(\sigma)$ *iff it is residually finite.*

Proof. As $\mathrm{Aut}(\sigma)$ is countable, every set of periodic points $\mathrm{Per}^0_n(X)$ of least period $n \in \mathbb{N}$ is finite by Theorem 4.6. For any automorphism $\varphi \in \mathrm{Aut}(\sigma) \setminus \{\mathrm{Id}_X\}$ there exists some $n \in \mathbb{N}$ with $\varphi|_{\mathrm{Per}^0_n(X)} \neq \mathrm{Id}_{\mathrm{Per}^0_n(X)}$ ($\mathrm{Per}(X)$ is a dense subset of X). The map $\alpha :\ \mathrm{Aut}(\sigma) \to \mathcal{S}_{\mathrm{Per}^0_n(X)}$ is thus a nontrivial homomorphism into the finite group $\mathcal{S}_{\mathrm{Per}^0_n(X)}$ with $\alpha(\varphi) \neq \mathrm{Id}_{\mathrm{Per}^0_n(X)}$. This proves $\mathrm{Aut}(\sigma)$ residually finite.

Assume $\mathrm{Aut}(\sigma)$ contains a nontrivial divisible group A. Now for every $\varphi \in A \setminus \{\mathrm{Id}_X\}$ one has a group-homomorphism $\alpha|_A :\ A \to H$ with H finite and $\varphi \notin \ker(\alpha)$. Denote by $k \in \mathbb{N}$ the cardinality of ${}^{A}/_{\ker(\alpha)} \leq H$. Since A is divisible there exists a k-th root $\psi \in A$ of φ. But then calculating in ${}^{A}/_{\ker(\alpha)}$ using cosets: $\ker(\alpha) = [\mathrm{Id}_X] = [\psi]^k = [\psi^k] = [\varphi]$ immediately yields the contradiction $\varphi \in \ker(\alpha)$.

Residual finiteness of some abstract group H always implies the non-existence of infinite simple subgroups inside H, since there are no nontrivial homomorphisms from infinite simple groups into finite groups.

Obviously non residually finite subgroups of ${}^{\mathbb{Q}}/_{\mathbb{Z}}$ cannot be subgroups of $\mathrm{Aut}(\sigma)$. As residually finite subgroups of ${}^{\mathbb{Q}}/_{\mathbb{Z}}$ are isomorphic to a countable direct sum $\bigoplus_{p \text{ prime}} H_p$ of finite groups $H_p \leq \mathbb{Z}(p^\infty)$, those can be embedded into $\mathrm{Aut}(\sigma)$. □

According to Theorem 4.6 the class of locally compact countable state Markov shifts with $\mathrm{Aut}(\sigma)$ countable admits exactly the same definition of sign- and gyration-homomorphisms as was introduced for SFTs in [**BK1**]. Therefore one can carry over the whole theory of how automorphisms act on periodic points ([**KR2**], [**KRW1**]; e.g. study the existence of sign-gyration-compatibility conditions) based on these representations of $\mathrm{Aut}(\sigma)$ to the setting of countable state Markov shifts with **(FMDP)**.

OPEN PROBLEM: After all these similarities between the automorphism groups of SFTs and countable state Markov shifts with property **(FMDP)** – both are countably infinite, residually finite groups with a seemingly equal subgroup structure, being discrete with respect to the compact-open topology and having the same center (see Section 5) – we may ask the question whether all countable automorphism groups that show up for transitive, locally compact, countable state Markov shifts are already realized for transitive SFTs. Unfortunately up to now we do not know of any property that distinguishes between the automorphism groups of these two subshift-classes. However one possible clue to decide this question might be the direct sum decomposition of $\mathrm{Aut}(\sigma)$ for a subclass of **(FMDP)**-Markov shifts presented in Section 6.

The results obtained so far already give a coarse classification of all transitive, countable state Markov shifts (X, σ) via their automorphism groups into 5 mutually disjoint, conjugacy-invariant classes:

	(X, σ) non locally compact	(X, σ) locally compact
$\mathrm{Aut}(\sigma)$ uncountable, non residually finite	very weak restrictions on algebraic properties and subgroups; e.g. subshifts satisfying Proposition 4.2	weak restrictions, due to the absense of a zetafunction and of **(FMDP)**; e.g. locally-compact subshifts satisfying Proposition 4.4
$\mathrm{Aut}(\sigma)$ uncountable, residually finite	no nontrivial divisible, no infinite simple subgroups; e.g. non locally compact, countable state Markov shifts with formal zetafunction	no nontrivial divisible, no infinite simple subgroups; examples can be constructed from graph presentations of transitive, locally compact, countable state Markov shifts with formal zetafunctions by doubling (n-folding) all edges
$\mathrm{Aut}(\sigma)$ countable, thus residually finite	not existent !	strong restrictions like in the SFT case; this class contains exactly the transitive, locally compact, countable state Markov shifts with **(FMDP)**

5. *Ryan's theorem for countable state Markov shifts*

As we have seen in the previous section, it is difficult to describe the automorphism groups of topological Markov shifts as abstract groups. Thus we look for further group-theoretic properties describing $\mathrm{Aut}(\sigma)$ and limiting the set of possible groups. One such property examined for SFTs is the center $\mathfrak{Z} = \mathfrak{Z}(\mathrm{Aut}(\sigma))$. J. Ryan ([**Rya1**] and [**Rya2**]) proved that for all transitive SFTs the center consists exactly of the powers of the shift map and is therefore (for all nontrivial, transitive SFTs) isomorphic to $\mathbb{Z}$.

Since by definition σ has to commute with every element in $\mathrm{Aut}(\sigma)$, we get $\{\sigma^i \mid i \in \mathbb{Z}\} \leq \mathfrak{Z}$ not just for Markov shifts but for any subshift (X, σ). Therefore the automorphism group of any nontrivial, transitive subshift has to have a center containing $\mathbb{Z}$ as a subgroup. Moreover the center is a normal subgroup in $\mathrm{Aut}(\sigma)$. This excludes certain abstract groups from being realized as automorphism groups of subshifts. For example:

PROPOSITION 5.1. *The automorphism group of any transitive, countable state Markov shift (nontrivial subshift) is not isomorphic to either $\mathcal{S}_{\mathbb{N}}$ or $\mathcal{S}_{\mathbb{N},\mathfrak{f}}$.*

Proof. Suppose $\mathrm{Aut}(\sigma) \cong \mathcal{S}_{\mathbb{N}}$. The theorem of J. Schreier and S. Ulam [**SU**] gives the Jordan-Hölder decomposition $\mathcal{S}_{\mathbb{N}} \rhd \mathcal{S}_{\mathbb{N},\mathfrak{f}} \rhd \mathcal{A}_{\mathbb{N},\mathfrak{f}} \rhd \{1\}$ (factor groups being simple) containing all normal subgroups of $\mathcal{S}_{\mathbb{N}}$. Therefore $\mathfrak{Z} \trianglelefteq \mathrm{Aut}(\sigma)$ has to be isomorphic to one of these four normal subgroups. $\mathcal{S}_{\mathbb{N}}$, $\mathcal{S}_{\mathbb{N},\mathfrak{f}}$, $\mathcal{A}_{\mathbb{N},\mathfrak{f}}$ are non-abelian groups, so they can be ruled out immediately. The remaining case $\mathfrak{Z} \cong \{1\}$ contradicts $\mathfrak{Z} \geq \{\sigma^i \mid i \in \mathbb{Z}\}$.

The same argument shows $\mathrm{Aut}(\sigma) \ncong \mathcal{S}_{\mathbb{N},\mathfrak{f}}$. □

After some preliminaries we can reprove Ryan's theorem for countable state Markov shifts:

LEMMA 5.2. *Every automorphism of some transitive Markov shift acting trivially on the set of periodic σ-orbits is a power of the shift map.*

Proof. It suffices to show that any automorphism $\varphi \in \mathrm{Aut}(\sigma)$ of the transitive Markov shift (X,σ) inducing the identity on the set of periodic σ-orbits just shifts all periodic points of large-enough period by a common amount. Since every point in X can be approximated by a sequence of periodic points of large period, this already fixes the action of φ on all of X and proves φ being some power of σ.

Choose a periodic point $x \in \mathcal{O}_1$ from some minimal σ-orbit $\mathcal{O}_1 \subseteq X$ and let $N_1 \in \mathbb{N}$ be the orbit length of $\mathcal{O}_1$. Then the block $l_1 := x_{[0,N_1)} \in \mathcal{B}_{N_1}(X)$ defines x and cannot overlap itself nontrivially. Now $\varphi(x) = \sigma^{s_1}(x)$ for $-\frac{1}{2}N_1 < s_1 \leq \frac{1}{2}N_1$ uniquely determined. As φ is continuous, mapping all finite σ-orbits onto itself, there is a coding length $n_1 \in \mathbb{N}$ such that $\big(\varphi(y)\big)_{[0,N_1)} = \big(\sigma^{s_1}(x)\big)_{[0,N_1)}$ for all $y \in {}_{-n_1N_1}[{l_1}^{2n_1}]$. Let $\mathcal{O}_2$, $\mathcal{O}_3$ be two distinct σ-orbits of lengths N_2, $N_3 \in \mathbb{N}$ larger than $2(n_1+1)N_1$ and let $l_2 \in \mathcal{B}_{N_2}(X)$, $l_3 \in \mathcal{B}_{N_3}(X)$ be defining blocks for $\mathcal{O}_2$, $\mathcal{O}_3$. Once again one has $\varphi({l_2}^{\infty}) = \sigma^{s_2}({l_2}^{\infty})$ and $\varphi({l_3}^{\infty}) = \sigma^{s_3}({l_3}^{\infty})$ for unique $-\frac{1}{2}N_2 < s_2 \leq \frac{1}{2}N_2$ and $-\frac{1}{2}N_3 < s_3 \leq \frac{1}{2}N_3$. Moreover there are numbers n_2, $n_3 \in \mathbb{N}$ for which $\big(\varphi(y)\big)_{[0,N_i)} = \big(\sigma^{s_i}({l_i}^{\infty})\big)_{[0,N_i)}$ whenever $y \in {}_{-n_iN_i}[{l_i}^{2n_i}]$ $(i := 2,3)$.

Using the irreducibility of X one can find blocks $p_{12}, p_{23}, p_{31} \in \mathcal{B}(X)$ of minimal length such that $l_1\, p_{12}\, l_2\, p_{23}\, l_3\, p_{31}\, l_1 \in \mathcal{B}(X)$ is admissible for X. For $m \in \mathbb{N}$ with $m\,N_1 > \max\{|l_3\, p_{31}\, l_1\, p_{12}\, l_2|\,, |l_2\, p_{23}\, l_3|\}$ define some periodic point $z := ({l_1}^{2n_1+m}\, p_{12}\, {l_2}^{2n_2}\, p_{23}\, {l_3}^{2n_3}\, p_{31})^{\infty} \in X$ which by construction has least period $M := (2n_1+m)N_1 + 2(n_2N_2 + n_3N_3) + |p_{12}\, p_{23}\, p_{31}|$. In particular a block ${l_1}^{m+1}$ can only occur inside ${l_1}^{2n_1+m}$. As before $\varphi(z) = \sigma^{s}(z)$ for $-\frac{1}{2}M < s \leq \frac{1}{2}M$ unique.

Now $\sigma^{n_1N_1}(z) \in {}_{-n_1N_1}[{l_1}^{2n_1+m}]$ implies

$$\big(\sigma^{s+n_1N_1}(z)\big)_{[0,a)} = \big(\varphi \circ \sigma^{n_1N_1}(z)\big)_{[0,a)} = \big(\varphi(x)\big)_{[0,a)} = \big(\sigma^{s_1}(x)\big)_{[0,a)}$$

with $a := (m+1)N_1$. Using the bound on s and the fact that ${l_1}^{m+1}$ has to be a subblock of ${l_1}^{2n_1+m}$ guarantees $-(n_1+1)N_1 < s \leq (n_1+1)N_1$. Since $N_2 > 2(n_1+1)N_1$, this implies $-\frac{1}{2}N_2 < s \leq \frac{1}{2}N_2$. But then $s = s_2$, because using the coding length of the block l_2 one gets $\big(\sigma^{s+(2n_1+m)N_1+|p_{12}|+n_2N_2}(z)\big)_{[0,N_2)} = \big(\varphi \circ \sigma^{(2n_1+m)N_1+|p_{12}|+n_2N_2}(z)\big)_{[0,N_2)} = \big(\varphi({l_2}^\infty)\big)_{[0,N_2)} = \big(\sigma^{s_2}({l_2}^\infty)\big)_{[0,N_2)}$. As $N_3 > 2(n_1+1)N_1$, the same argument shows $s = s_3$. Therefore all periodic σ-orbits of length greater than $2(n_1+1)N_1$ are shifted under φ by the same amount $s = s_2 = s_3$. □

Remark: The proof of Lemma 5.2 merely relies on the Markov property and the transitivity, but not on the cardinality of the alphabet. Hence it can be used for SFTs, countable state (and even larger) Markov shifts.

Using the periodic-orbit representation of the automorphism group, originally introduced for SFTs by M. Boyle and W. Krieger (see [**BK1**] or [**BLR**]), we can translate Lemma 5.2 into the language of group theory. To achieve this we define the periodic-orbit representation ρ for automorphism groups of (countable state) Markov shifts exactly as for SFTs:

Let $\mathrm{Orb}_n(X) := {}^{\mathrm{Per}^0_n(X)}/_{\langle\sigma\rangle}$ the set of σ-orbits of length $n \in \mathbb{N}$. Then

$$\rho : \mathrm{Aut}(\sigma) \to \prod_{n=1}^{\infty} \mathrm{Aut}\big(\mathrm{Orb}_n(X), \sigma\big), \quad \varphi \mapsto \rho(\varphi) := \big(\rho_n(\varphi)\big)_{n\in\mathbb{N}}$$

where $\rho_n(\varphi) \in \mathrm{Aut}\big(\mathrm{Orb}_n(X), \sigma\big)$ is the permutation on the set of σ-orbits $\mathrm{Orb}_n(X)$ induced by $\varphi|_{\mathrm{Per}^0_n(X)}$. $\rho_n(\varphi)$ is well-defined, since $\varphi|_{\mathrm{Per}^0_n(X)} \in \mathrm{Aut}\,\big(\mathrm{Per}^0_n(X), \sigma\big)$ commutes with $\sigma|_{\mathrm{Per}^0_n(X)}$ and $\rho_n(\sigma) = \mathrm{Id}_{\mathrm{Orb}_n(X)}$.

Corollary 5.3. *For every transitive Markov shift* (X,σ) *the periodic-orbit representation of* $\mathrm{Aut}(\sigma)$ *is faithful on the group* ${}^{\mathrm{Aut}(\sigma)}/_{\langle\sigma\rangle}$.

Proof. Let $\varphi \in \mathrm{Aut}(\sigma)$ with $\rho(\varphi) = \mathrm{Id}$. Then $\rho_n(\varphi) = \mathrm{Id}_{\mathrm{Orb}_n(X)}$ for all $n \geq 1$. Lemma 5.2 implies $\varphi \in \langle\sigma\rangle$, hence $\varphi \in [\mathrm{Id}] \in {}^{\mathrm{Aut}(\sigma)}/_{\langle\sigma\rangle}$. □

Theorem 5.4. *The center of the automorphism group of any transitive (countable state) Markov shift consists exactly of the powers of the shift map.*

Proof. Suppose the automorphism $\varphi \in \mathrm{Aut}(\sigma)$ is no power of the shift map. Following from Lemma 5.2 there are two distinct periodic σ-orbits $\mathcal{O}_1$, $\mathcal{O}_2$ of some length $N \in \mathbb{N}$ such that $\varphi(\mathcal{O}_1) = \mathcal{O}_2$. Let $x^{(1)} \in \mathcal{O}_1$, $x^{(2)} := \varphi(x^{(1)}) \in \mathcal{O}_2$ and $x^{(3)} := \varphi(x^{(2)}) \in \mathrm{Per}^0_N(X)$. By continuity, the blocks $l_i := (x^{(i)})_{[0,N)} \in \mathcal{B}_N(X)$ $(i := 1,2,3)$ satisfy $\varphi({}_{-m_1N}[{l_1}^{2m_1}]) \subseteq {}_0[l_2]$ und $\varphi({}_{-m_2N}[{l_2}^{2m_2}]) \subseteq {}_0[l_3]$ for m_1, $m_2 \in \mathbb{N}$ large enough.

Define a periodic point $y^{(1)} := ({l_1}^{m+m_1}\, p_{12}\, {l_2}^{m_2}\, p_{21}\, {l_1}^{m_3}\, \tilde{p}_{12}\, {l_2}^{m_4}\, \tilde{p}_{21}\, {l_1}^{m_5})^\infty \in X$ of large period, where p_{12}, p_{21}, $\tilde{p}_{12}$, $\tilde{p}_{21} \in \mathcal{B}(X)$ are non empty blocks not containing a complete block l_1, l_2 or l_3 (those exist, since X is irreducible). Furthermore $m_i \in \mathbb{N}$ $(1 \leq i \leq 5)$ are chosen large enough to get an image of the form

$y^{(2)} := \varphi(y^{(1)}) = ({l_2}^{m+n_1}\, q_{23}\, {l_3}^{n_2}\, q_{32}\, {l_2}^{n_3}\, \tilde{q}_{23}\, {l_3}^{n_4}\, \tilde{q}_{32}\, {l_2}^{n_5})^\infty$ with $n_i \geq 1$ $(1 \leq i \leq 5)$. As suggested by this representation no prefix of q_{23}, $\tilde{q}_{23}$ and no suffix of q_{32}, $\tilde{q}_{32}$ shall be a complete block l_2; no prefix of q_{32}, $\tilde{q}_{32}$ and no suffix of q_{23}, $\tilde{q}_{23}$ is a complete block l_3. Finally m_2, m_4 can be tuned to get $n_2 \neq n_4$ and $m \in \mathbb{N}$ should satisfy $m\,N > (m_1 + m_2 + m_3 + m_4 + m_5)\,N + |p_{12}\,p_{21}\,\tilde{p}_{12}\,\tilde{p}_{21}|$. This guarantees $y^{(1)}$, $y^{(2)} \in \mathrm{Per}^0_M(X)$ with $M := |p_{12}\,p_{21}\,\tilde{p}_{12}\,\tilde{p}_{21}| + N\,(m+m_1+m_2+m_3+m_4+m_5)$; the block $(y^{(2)})_{[0,M)}$ has only trivial overlaps (consisting of blocks ${l_2}^i$) to itself.

Define an involutoric sliding-block-code $\psi:\ X \to X$ interchanging the blocks ${l_2}^{m+n_1}\, q_{23}\, {l_3}^{n_2}\, q_{32}\, {l_2}^{n_3}\, \tilde{q}_{23}\, {l_3}^{n_4}\, \tilde{q}_{32}\, {l_2}^{n_5}$ and ${l_2}^{m+n_1}\, q_{23}\, {l_3}^{n_4}\, q_{32}\, {l_2}^{n_3}\, \tilde{q}_{23}\, {l_3}^{n_2}\, \tilde{q}_{32}\, {l_2}^{n_5}$; ${l_2}^{m+n_1}\, q_{23}$ and $\tilde{q}_{32}\, {l_2}^{n_5}$ acting as markers. So $\psi(y^{(1)}) = y^{(1)}$, but $\psi(y^{(2)}) \neq y^{(2)}$. Obviously $\psi \in \mathrm{Aut}(\sigma)$ does not commute with φ, since $(\varphi \circ \psi)(y^{(1)}) = \varphi(y^{(1)}) = y^{(2)} \neq \psi(y^{(2)}) = (\psi \circ \varphi)(y^{(1)})$. So $\varphi \notin \mathfrak{Z}$ and $\mathfrak{Z} = \{\sigma^i \mid i \in \mathbb{Z}\}$. □

Concerning the center of $\mathrm{Aut}(\sigma)$ we get the same restrictions for (countable state) Markov shifts as for SFTs. Theorem 5.4 additionally can be used to exclude further abstract groups (with center non-isomorphic to $\mathbb{Z}$) from being realized as automorphism groups of transitive Markov shifts. In particular the comparison between Markov shifts and coded systems yields considerably differences. Recalling some of the results by D. Fiebig and U.-R. Fiebig presented in [**FF2**], there are coded systems with automorphism groups isomorphic to any infinite, finitely generated abelian group (e.g. $\mathrm{Aut}(\sigma) \cong \langle\sigma\rangle \oplus \mathbb{Z}$) as well as isomorphic to $\langle\sigma\rangle \oplus \mathbb{Z}[1/2]$ or $\langle\sigma\rangle \oplus G$ with $G \leq \mathbb{Q}/\mathbb{Z}$ residually finite. Of course non of these can occur for transitive Markov shifts. Another result from [**FF2**] proves the existence of a coded system (X, σ_X) with automorphism group $\mathrm{Aut}(\sigma_X) \cong \langle\sigma_X\rangle \oplus \mathrm{Aut}(\sigma_Y)$, where (Y, σ_Y) is any nontrivial subshift with periodic points dense – a situation completely impossible for transitive Markov shifts, unless $\mathfrak{Z}(\mathrm{Aut}(\sigma_Y))$ is trivial, which is only true for a system of fixed points.

6. *Direct sum decomposition of* Aut(σ)

The research carried out and published by D. Fiebig and U.-R. Fiebig in [**FF2**] proves that the automorphism groups of coded systems often split into a direct sum of the cyclic group generated by the shift map and a second group which can vary – depending on the coded system – in a large set of abstract groups. Moreover their method is highly constructive.

We will show the same phenomenon for the class of transitive, locally compact, countable state Markov shifts that can be presented as edge shifts on thinned-out graphs:

Definition 6.1. *A strongly connected, locally finite directed graph $G = (V, E)$ with $|E| = \aleph_0$ is a thinned-out graph, iff it contains a vertex $v \in V$ such that the set $L := \{l_n \mid n \in \mathbb{N}\}$ of first-return-loops at v satisfies:*

$$\forall\, M \in \mathbb{N}_0\ \exists\, N \in \mathbb{N}:\ \forall\, n \geq N:\ |l_{n+1}| - |l_n| > M \tag{GC}$$

Remark: The term 'thinned-out' has been chosen, since the gaps in the sequence of lengths of the first-return-loops at v grow unbounded (the further from v, the thinner the structure of G). For any given bound M there are at most finitely many first-return-loops at v with length-difference less or equal to M. In particular the growth condition (GC) for $M = 0$ implies the existence of at most finitely many pairs of first-return-loops at v having a common length.

The following two propositions expose some general properties of thinned-out graphs. Proposition 6.2 proves v to be already a (one-element) vertex-ROME for G, whereas Proposition 6.3 shows that the thinned-out graphs form a proper subclass of the graphs having **(FMDP)**.

As usual a path/loop is called simple, if it has no proper closed subpath.

PROPOSITION 6.2. *Let $G = (V, E)$ be some thinned-out graph. The set of first-return-loops at a vertex $v \in V$ satisfies* (GC)*, iff v is part of any bi-infinite walk along the edges of G. All the first-return-loops at such a vertex v are simple and v shows up in every non-simple path in G.*
Neither does G contain two vertex-disjoint loops nor two vertex-disjoint, bi-infinite walks.

Proof. "$\Longrightarrow$": Suppose for every vertex $v \in V$ there is a loop $l_v := e_1\, e_2 \ldots e_{|l_v|}$ ($e_i \in E$) avoiding v, i.e. $\mathfrak{t}(e_{|l_v|}) = \mathfrak{i}(e_1) \neq v$ and $\forall\, 1 \leq i < |l_v| :\ \mathfrak{t}(e_i) = \mathfrak{i}(e_{i+1}) \neq v$. Since G is strongly connected one can choose a shortest path p from v to $\mathfrak{i}(e_1)$ and a shortest path q from $\mathfrak{t}(e_{|l_v|})$ back to v. The subset $\left\{p\, {l_v}^i\, q \;\middle|\; i \in \mathbb{N}_0\right\}$ of first-return-loops at v contradicts the growth condition (GC) for $M := |l_v|$. As G is thinned-out, it contains at least one vertex that is part of every loop.

Suppose there is a bi-infinite, simple walk $w := \ldots w_{-3}\, w_{-2}\, w_{-1} \,.\, w_0\, w_1\, w_2 \ldots$ in G ($w_i \in E$, $\mathfrak{t}(w_i) = \mathfrak{i}(w_{i+1}) \neq v\ \forall\, i \in \mathbb{Z}$) avoiding v. Let p_1 be some shortest path connecting v to $\mathfrak{i}(w_0)$. Fix a shortest path p_2 from v to $\mathfrak{i}(w_{-|p_1|})$ and choose an infinite sequence $(q_i)_{i\in\mathbb{N}}$ of shortest paths connecting $\mathfrak{t}(w_{n_i})$ with $n_i := \sum_{j=1}^{i-1} |q_j|$ back to v. All these paths are non-empty and distinct. Once more they yield an infinite number of pairs $p_1\, w_0 \ldots w_{n_i}\, q_i,\ p_2\, w_{-|p_1|} \ldots w_0 \ldots w_{n_i}\, q_i$ ($i \in \mathbb{N}$) of first-return-loops at v, that violate (GC) for $M := |p_2|$. Therefore any vertex that is not part of every bi-infinite walk in G does not fulfill (GC).

"$\Longleftarrow$": Let $v \in V$ be contained in any bi-infinite walk along the edges of G and choose $w \in V$ such that the first-return-loops at w satisfy (GC). For $w = v$ the statement is obviously true. Assume $w \neq v$. Using the previous part of the proof w shows up in every bi-infinite walk as well. In particular w is part of any loop in G and the sequence of lengths of first-return-loops at w and at v agree with each other. Actually there is either only one simple path leading from w to v or only one simple path leading from v to w. The first-return-loops at v are cyclically permuted first-return-loops at w and vice versa.

The remaining statements follow immediately. □

PROPOSITION 6.3. *Every thinned-out graph has the property* **(FMDP)**.

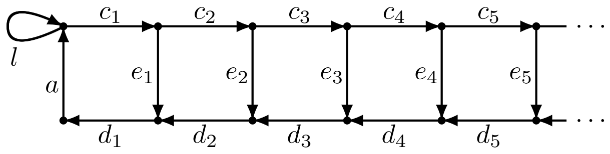


Figure 2. Graph presentation of a transitive, locally compact, countable state Markov shift with **(FMDP)**. The vertex $\mathfrak{t}(a)$ is a ROME; every bi-infinite walk contains at least one of the edges labeled a or l.

Proof. Let $G = (V, E)$ be a thinned-out graph. Assume the set $L := \{l_n \mid n \in \mathbb{N}\}$ of first-return-loops at $v \in V$ satisfies the growth condition (GC). As pointed out in the remark following Definition 6.1 the sequence $(|l_n|)_{n \geq N}$ is strictly increasing after some bound $N \in \mathbb{N}$. Obviously the finite subset $F := \{e \in E \mid \mathfrak{i}(e) = v\} \cup \{e \in E \mid \exists\, n \leq N : \ e \in l_n\}$ of edges covers all double paths in G. According to Lemma 3.1 this proves G to be a **(FMDP)**-graph. □

Figure 2 shows an example of a graph with property **(FMDP)** that is not thinned-out. The lengths of the first-return-loops at $\mathfrak{t}(a)$ are $1, 4, 6, 8, 10, \ldots,$ whereas for any other vertex this sequence comprises all large enough natural numbers.

Obviously Definition 6.1 is a priori not invariant under (topological) conjugacy in the sense that given a transitive, locally compact, countable state Markov shift (X_G, σ) defined on a thinned-out graph G we can – using a finite number of state splittings – easily construct another graph presentation that contains vertex-disjoint loops and is therefore no longer thinned-out.

We overcome this technicality by calling a transitive, locally compact, countable state Markov shift (X, σ) thinned-out, iff its set of subshift presentations $\text{Pres}(X)$ contains an edge shift on some thinned-out graph. Via this little detour we define a conjugacy invariant subclass of all countable state Markov shifts.

The reason for studying this class is the rigid structure of a thinned-out graph G forcing each automorphism to map the set of σ-orbits corresponding to bi-infinite, simple walks along the edges of G onto itself. Usually this need not be the case even for transitive, countable state Markov shifts with **(FMDP)**. We illustrate this for the edge shift (X, σ) on the **(FMDP)**-graph displayed in Figure 2. There is an order 2 automorphism $\varphi : \ X \to X$ that scanning a point $x \in X$ replaces every block $a\, l\, l\, c_1 \ldots c_n\, e_n$ with $a\, c_1 \ldots c_{n+1}\, e_{n+1}\, d_{n+1}$ $(\forall\, n \in \mathbb{N})$ and vice versa. By continuity φ maps the point $y := \ldots d_3\, d_2\, d_1 \,.\, a\, c_1\, c_2\, c_3 \ldots \in X$ that corresponds to a bi-infinite, simple walk into $\varphi(y) = \ldots d_3\, d_2\, d_1 \,.\, a\, l\, l\, c_1\, c_2\, c_3 \ldots \in X$, a point that does not correspond to any bi-infinite, simple walk in G.

Automorphisms of topological Markov shifts seem not to distinguish between bi-infinite walks and bi-infinite, simple walks as long as the minimal differences between the lengths of first-return-loops remain bounded. Whereas for thinned-out graphs (unbounded length-differences) they have to respect simple walks, as we will see below. This property can be used to show that any automorphism acts on all points avoiding a certain finite set of edges like a power of the shift map. Moreover this set can always be chosen from the complement of all bi-infinite, simple walks. Therefore one can factor out the cyclic group generated by σ and represent any automorphism as a composition of a power of the shift with an automorphism which is the identity on all points corresponding to bi-infinite, simple walks. This gives the desired direct sum decomposition of $\text{Aut}(\sigma)$ for edge shifts on thinned-out graphs. As the existence of such a decomposition is a purely group theoretical property of the conjugacy invariant automorphism group, the result holds – independently of the chosen presentation – for all thinned-out Markov shifts.

THEOREM 6.4. *Let (X, σ) be some thinned-out Markov shift. Any automorphism acts on the set of σ-orbits corresponding to bi-infinite, simple walks in any graph presentation on some thinned-out graph like a power of the shift map.*
The automorphism group $\text{Aut}(\sigma)$ splits into the direct sum of the cyclic group generated by σ and another countably infinite, centerless group. If (X, σ) is presented on a thinned-out graph $G = (V, E)$ one gets:

$$\text{Aut}(\sigma) \cong \langle\sigma\rangle \oplus \big\{\varphi \in \text{Aut}(\sigma) \,\big|\, \exists\, K_\varphi \subsetneq E \text{ finite: } K_\varphi \text{ does not contain any edge from a bi-infinite, simple walk on } G \;\wedge\; \varphi|_{\text{Orb}(K_\varphi)^{\complement}} = \text{Id}_{\text{Orb}(K_\varphi)^{\complement}} \big\}$$

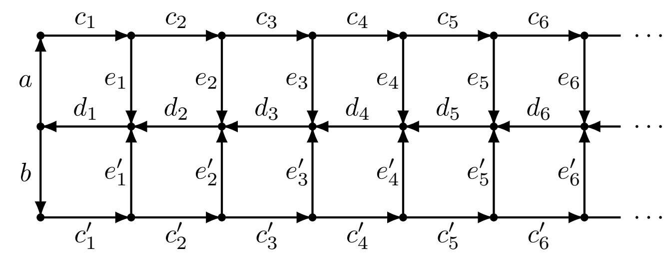
where $\text{Orb}(K_\varphi)^{\complement} := X \setminus \bigcup_{n\in\mathbb{Z}} \sigma^n \big(\bigcup_{k\in K_\varphi} {}_0[k]\big)$ *denotes the orbit-complement of* K_φ.

We postpone the proof of Theorem 6.4 to the last section of this paper. Instead to round off our work we describe some thinned-out graph presentations. Theorem 6.4 can be applied directly to the corresponding subshifts giving lots of examples of transitive, locally compact, countable state Markov shifts with countably infinite automorphism groups being a direct sum.

First look at the graph G displayed in Figure 2. Let $S : \mathbb{N} \to \mathbb{N}$ be some superlinear function, i.e. S is monotone and grows eventually faster than any linear function (like $S(n) := n^2$, $S(n) := 2^n$ etc.). Deleting any edge e_i with $i \notin S(\mathbb{N})$ from G gives a thinned-out graph.

Figure 3 shows two strongly connected, locally finite, countable graphs. Once more let $S : \mathbb{N} \to \mathbb{N}$ be a superlinear function. Remove all edges e_i, e'_j with $i \notin S(2\mathbb{N})$ and $j \notin S(2\mathbb{N} + 1)$ to get thinned-out graphs with more than one bi-infinite, simple walk (Figure 3, top graph) or even with a canonical boundary (for an exact definition of this term see [**FF1**]) consisting of more than one orbit (Figure 3, bottom graph).

More complicated thinned-out graphs can be constructed by identifying the vertex-ROMEs of two or more appropriate thinned-out graphs with each others: For instance merge two copies $G = (V, E)$ and $G' = (V', E')$ of the graph from Figure 2 by identifying the vertices $\mathfrak{t}(a) \in V$ and $\mathfrak{t}(a') \in V'$ and remove all edges $e_i \in E$ with $i \notin S(2\mathbb{N})$ as well as $e'_j \in E'$ with $j \notin S(2\mathbb{N} + 1)$ for $S : \mathbb{N} \to \mathbb{N}$ superlinear.



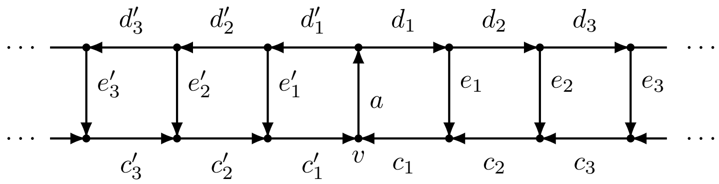


Figure 3. Graph presentations of two transitive, locally compact, countable state Markov shifts. The top graph has two bi-infinite, simple walks whereas the bottom graph contains four bi-infinite, simple walks and the canonical boundary of its Markov shift consists of two orbits.

7. *Proof of Theorem 6.4*

This final section is primarily dedicated to the proof of Theorem 6.4, even so it comprises two more propositions (7.2 and 7.3) on the structure of strongly connected graphs with an one-element vertex-ROME, which may be of general interest.

At first recall that every topological conjugacy $\gamma: X \to Y$ (in particular every automorphism) between two locally compact, countable state Markov shifts (X, σ_X), (Y, σ_Y) has some kind of local coding-lengths:

$$\forall\, C \subsetneq Y \text{ cylinder set } \exists\, s_C \le t_C \in \mathbb{Z}: \qquad x = (x_i)_{i\in\mathbb{Z}} \in \gamma^{-1}(C) \iff {}_{s_C}[x_{s_C} \dots x_{t_C}] \subseteq \gamma^{-1}(C) \qquad \text{(CL)}$$

In other words $\gamma^{-1}(C) = \bigcup_{x\in\gamma^{-1}(C)} {}_{s_C}[x_{s_C} \dots x_{t_C}]$ and is thus presentable as a finite union of cylinder sets. The easy argument for this is that $C \subsetneq Y$ compact-open immediately forces $\gamma^{-1}(C)$ to be compact-open.

We start the proof of Theorem 6.4 with a lemma pointing out that any automorphism not only respects the σ-orbits of bi-infinite, simple walks on a thinned-out graph but that all corresponding points are just shifted by an uniform amount. This is due to the unbounded growth of the length-differences in the sequence of first-return-loops at an one-element vertex-ROME. The structure of the graph far away from this single vertex has thus a large influence on the (non-)existence of certain automorphisms.

Lemma 7.1. *Let (X_G, σ) be an edge shift on some thinned-out graph $G = (V, E)$. Any automorphism $\varphi \in \operatorname{Aut}(\sigma)$ induces some permutation on the bi-infinite, simple walks along the edges of G.*
Moreover φ acts on corresponding points like a fixed power of the shift map and so the permutation induced by φ is actually the identity.

Proof. We build up a whole string of arguments based on the local compactness of X_G and the validity of the growth condition (GC) for the set L of first-return-loops at some vertex $v \in V$. Let $F := \{e \in E \mid \mathfrak{i}(e) = v\}$ the finite set of out-going edges at v, then F constitutes a finite edge-ROME in G.

In the following we do not distinguish between bi-infinite walks on G and σ-orbits of corresponding points in X_G. As φ is a shiftcommuting bijection it induces an injective mapping $\tilde{\varphi} : \operatorname{Orb}(X_G) \to \operatorname{Orb}(X_G)$, $\operatorname{Orb}(x) \mapsto \operatorname{Orb}(\varphi(x))$ on the σ-orbits/walks in the obvious way. To prove the first statement of Lemma 7.1 one has to show that – by choice of F – any automorphism φ maps all representatives $x := \ldots x_{-3}\, x_{-2}\, x_{-1} \,.\, x_0\, x_1\, x_2 \ldots \in \operatorname{Orb}(x)$ of bi-infinite, simple walks with $x_0 \in F$ to representatives of simple walks on G. This is done in the next two claims.

Thereafter surjectivity and thus bijectivity of $\tilde{\varphi}$ restricted to the simple walks in G immediately follows from the existence of the inverse $\tilde{\varphi}^{-1}$ defined via $\tilde{\varphi}^{-1}\big(\operatorname{Orb}(x)\big) := \operatorname{Orb}\big(\varphi^{-1}(x)\big)$, which itself acts on the bi-infinite, simple walks.

Claim 1. *Let $K \subsetneq E$ be any finite subset of edges. The distance from an edge in K to the nearest edge in F is uniformly bounded for all points in X_G, i.e. there exists some bound $I \in \mathbb{N}$ independent of $y \in \bigcup_{k \in K} {}_0[k] \subsetneq X_G$ such that the block $y_{[-I,I]}$ does contain an element from F.*

Proof. As long as $B := \big\{y_{[-i,i]} \;\big|\; y \in \bigcup_{k \in K} {}_0[k] \;\wedge\; i \in \mathbb{N} \;\wedge\; \forall\, |j| \le i : \; y_j \notin F\big\}$ is finite (or empty) the statement is trivial. Otherwise at least one subset $B_k := \big\{y_{[-i,i]} \;\big|\; y \in {}_0[k] \;\wedge\; i \in \mathbb{N} \;\wedge\; \forall\, |j| \le i : \; y_j \notin F\big\}$ for some $k \in K$ has to be infinite. In which case using the local finiteness of G one could inductively choose a sequence of admissible blocks $\big(b^{(n)}\big)_{n \in \mathbb{N}_0}$ with $b^{(0)} := k$, $b^{(n+1)} := a_n\, b^{(n)}\, c_n$ $(a_n, c_n \in E)$ such that $\big(B_{b^{(n)}} := \big\{y_{[-i,i]} \;\big|\; y \in {}_{-n}[b^{(n)}] \;\wedge\; i > n \;\wedge\; \forall\, |j| \le i : \; y_j \notin F\big\}\big)_{n \in \mathbb{N}_0}$ is a decreasing nested sequence of infinite sets. Since X_G is closed this would imply the existence of a point $x \in X_G$ with $x_{[-i,i]} \in B_{b^{(i)}}$ $\forall\, i \in \mathbb{N}_0$. x completely avoids all edges in F and thus contradicts v being a vertex-ROME for G. □

Since X_G is locally compact the finite union of all zero-cylinders ${}_0[f]$ with $f \in F$ is compact-open. So for any $\varphi \in \operatorname{Aut}(\sigma)$ one can find a finite subset $K \subsetneq E$ of edges (depending on φ) with

$$F \subseteq K \quad \text{and} \quad \varphi^{-1}\big(\textstyle\bigcup_{f \in F} {}_0[f]\big) \subseteq \textstyle\bigcup_{k \in K} {}_0[k] \qquad \text{(FP)}$$

Using Claim 1 for a set $K \subsetneq E$ as assumed in (FP) shows that all representatives $x \in X_G$ of simple walks with $x_0 \in F$ do not contain any edges from K outside the central block $x_{[-I,I]}$. So neither the left-infinite ray $\big(\varphi(x)\big)_{(-\infty,-I)}$ nor the right-infinite ray $\big(\varphi(x)\big)_{(I,\infty)}$ of the image of x under φ contains an edge from F.

The vertex-ROME v shows up particularly in every loop in G. So the simple walks are characterized by the property of containing precisely one edge starting at v. Therefore φ acts on the representatives of bi-infinite, simple walks in G iff

CLAIM 2. *The central block* $\big(\varphi(x)\big)_{[-I,I]}$ *contains exactly one edge from* F.

Proof. According to (CL) the finite block $\big(\varphi(x)\big)_{[-I,I]}$ has bounded coding-length, that is, there exists some $J \geq I$ such that $\big(\varphi(y)\big)_{[-I,I]} = \big(\varphi(x)\big)_{[-I,I]}$ for all $y \in {}_{-J}[x_{-J} \dots x_J]$.

Construct a special point $y := x_{(-\infty,-1]} \,.\, x_{[0,J]} \underbrace{?\,?\,?} x_{[-J,-1]}\, x_{[0,\infty)} \in X_G$ where the central block $x_{[0,J]} \underbrace{?\,?\,?} x_{[-J,-1]}$ consists of precisely two first-return-loops $l_m, l_n \in L$ at v chosen by the following procedure:

W.l.o.g. let the elements in L be sorted according to their length, i.e. $|l_i| \leq |l_j|$ whenever $i \leq j$. The validity of (GC) for L implies the existence of $N_1 \in \mathbb{N}$ such that $|l_{N_1}| > 2J$ and $\big||l_i| - |l_j|\big| > 2I$ for all $i \geq N_1$ and $j \neq i$. Choose $l_m \in L$ to be the shortest first-return-loop starting with $x_{[0,|l_{N_1}|)}$. Let $N_2 \in \mathbb{N}$ with $|l_{N_2}| > |l_m|$ and $\big||l_i| - |l_j|\big| > |l_m| + 2I$ for all $i \geq N_2$ and $j \neq i$. Finally take $l_n \in L$ minimal ending in $x_{[-|l_{N_2}|,-1]}$ and set $x_{[0,J]} \underbrace{?\,?\,?} x_{[-J,-1]} := l_m\, l_n$.

As we have shown above $\big(\varphi(x)\big)_{[-I,I]}$ has to contain at least one edge from the set F (v is a vertex-ROME; there is no edge from F in $\big(\varphi(x)\big)_{(-\infty,-I)}$ or $\big(\varphi(x)\big)_{(I,\infty)}$). Therefore define $a := \max\big\{i \in \mathbb{Z} \bigm| -I \leq i \leq I \,\wedge\, \big(\varphi(x)\big)_i \in F\big\}$ and $b := \min\big\{i \in \mathbb{Z} \bigm| -I \leq i \leq I \,\wedge\, \big(\varphi(x)\big)_i \in F\big\}$.

Since $y_{(-\infty,J]} = x_{(-\infty,J]}$, $y_{[|l_m\, l_n|-J,\infty)} = x_{[-J,\infty)}$ and $2J+1$ is a coding-length for $\big(\varphi(x)\big)_{[-I,I]}$ one gets $\big(\varphi(y)\big)_{[-I,I]} = \big(\varphi(y)\big)_{[|l_m\, l_n|-I,|l_m\, l_n|+I]} = \big(\varphi(x)\big)_{[-I,I]}$ and the block $B := \big(\varphi(y)\big)_{[a,|l_m\, l_n|+b)}$ consists of a concatenation of elements from L.

Suppose $B = l_j \in L$ is a single first-return-loop, to get an immediate contradiction:

$$\big||l_n| - |l_j|\big| = \big||l_n| - |l_m\, l_n| - b + a\big| = \big||l_m| + b - a\big| \leq |l_m| + |b-a| \leq |l_m| + 2I$$

By choice of l_n this estimate would imply $l_j = l_n$, but $|l_m| > 2J \geq 2I$ and $-2I \leq b - a \leq 0$ results in $|l_j| = |l_m| + |l_n| + b - a > |l_n|$.

The equivalence $y_i \in F \iff i \in \{0, |l_m|, |l_m\, l_n|\}$ together with Claim 1 yields:

$$\forall\, I < i < |l_m| - I :\; y_i \notin K \quad \text{ as well as } \quad \forall\, |l_m| + I < i < |l_m\, l_n| - I :\; y_i \notin K$$

Consequently the appropriate restrictions on the image point are:

$$\forall\, a < i < |l_m| - I :\; \big(\varphi(y)\big)_i \notin F \quad \text{ and } \quad \forall\, |l_m| + I < i < |l_m\, l_n| + b :\; \big(\varphi(y)\big)_i \notin F$$

Now if B is assumed to start with a loop $l_i \in L$ and end in $l_j \in L$ the above gives: $|l_m| - I - a \leq |l_i| \leq |l_m| + I - a$, which can be transformed into $\big||l_m| - |l_i|\big| \leq 2I$, forcing $l_i = l_m$. In the same manner $|l_m\, l_n| + b - |l_m| - I \leq |l_j| \leq |l_m\, l_n| + b - |l_m| + I$ can be manipulated into $\big||l_n| - |l_j|\big| \leq 2I$, forcing $l_j = l_n$. Finally use $b \leq a$ to establish an upper bound on the length of B: $|B| = |l_m\, l_n| + b - a \leq |l_m| + |l_n|$.

This shows $B = l_m\, l_n$ and $a = b$. So $\big(\varphi(x)\big)_{[-I,I]}$ contains precisely one edge from F and φ induces some permutation on the set of bi-infinite, simple walks. □

We slightly generalize the idea from the proof of Claim 2 to show the remaining statements:

Claim 3. *The unique edge from F inside $\big(\varphi(x)\big)_{[-I,I]}$ is located at a common coordinate $-M_\varphi$ ($-I \leq M_\varphi \leq I$) for all representatives $x \in X_G$ of bi-infinite, simple walks with $x_0 \in F$.*

Proof. Let $x^{(1)}, x^{(2)} \in X_G$ be representatives of two distinct bi-infinite, simple walks on G with $x^{(1)}_0, x^{(2)}_0 \in F$ and denote by $a_1, a_2 \in \{-I, -I+1, \ldots, I\}$ the coordinates of the unique edge from F in $\big(\varphi(x^{(1)})\big)_{[-I,I]}$, $\big(\varphi(x^{(2)})\big)_{[-I,I]}$ respectively. W.l.o.g. let $a_1 \geq a_2$.

As above construct a point $y := x^{(1)}_{(-\infty,-1]} \,.\, l_m\, l_n\, x^{(2)}_{[0,\infty)} \in X_G$ with $l_m, l_n \in L$ such that l_m starts with $x^{(1)}_{[0,|l_{N_1}|)}$ and l_n ends in $x^{(2)}_{[-|l_{N_2}|,-1]}$, where $N_1, N_2 \in \mathbb{N}$ are chosen as in the previous proof (Here $J \geq I$ is a common coding-length for $\big(\varphi(x^{(1)})\big)_{[-I,I]}$ and $\big(\varphi(x^{(2)})\big)_{[-I,I]}$). Substituting $a := a_1$ and $b := a_2$ the remaining proof carries over directely from Claim 2 and $a_1 = a_2$. □

Since φ commutes with the shift map one instantly gets the equivalence $x_i \in F$ iff $\big(\varphi(x)\big)_{i-M_\varphi} \in F$ for any $i \in \mathbb{Z}$ and any representative $x \in X_G$ of some bi-infinite, simple walk.

Claim 4. *φ maps any representative $x \in X_G$ of some bi-infinite, simple walk on G into $\operatorname{Orb}(x)$, i.e. $\tilde{\varphi}$ is the identity.*

Proof. Let $J \geq I$ be a coding-length for $\big(\varphi(x)\big)_{[-I,I]}$ and define $N_1 \in \mathbb{N}$ as before. Construct a sequence of points $\big(y^{(k)} := x_{(-\infty,-1]}\, l^{(k)}_m\, l^{(k)}_n \,.\, l^{(k)}_m\, l^{(k)}_n\, x_{[0,\infty)} \in X_G\big)_{k\in\mathbb{N}}$ converging to x: Choose shortest first-return-loops $l^{(k)}_m$ starting with $x_{[0,|l_{N_1}|+k)}$. For every $k \in \mathbb{N}$ fix $N^{(k)}_2 \in \mathbb{N}$ such that $\big|l_{N^{(k)}_2}\big| > \big|l^{(k)}_m\big|$ and $\big||l_i| - |l_j|\big| > \big|l^{(k)}_m\big| + 2I$ for all $i \geq N^{(k)}_2$ and $j \neq i$ exactly as above. $l^{(k)}_n$ be the shortest element in L ending in $x_{[-|l_{N^{(k)}_2}|,-1]}$. Since $y^{(k)}_{[-|l_{N^{(k)}_2}|,|l_{N_1}|+k]} = x_{[-|l_{N^{(k)}_2}|,|l_{N_1}|+k]}$ and $\big|l_{N^{(k)}_2}\big| > \big|l_{N_1}\big| + k$ this procedure forces the convergence $y^{(k)} \xrightarrow{k\to\infty} x$.

The image points $\varphi(y^{(k)})$ satisfy:

$$\forall\, k \in \mathbb{N}: \quad \big(\varphi(y^{(k)})\big)_{[-I-|l^{(k)}_m\, l^{(k)}_n|,\, I-|l^{(k)}_m\, l^{(k)}_n|]} = \big(\varphi(y^{(k)})\big)_{[-I,I]} = \\ = \big(\varphi(y^{(k)})\big)_{[-I+|l^{(k)}_m\, l^{(k)}_n|,\, I+|l^{(k)}_m\, l^{(k)}_n|]} = \big(\varphi(x)\big)_{[-I,I]}$$

so the blocks $\big(\varphi(y^{(k)})\big)_{[-M_\varphi-|l^{(k)}_m\, l^{(k)}_n|,\, -M_\varphi)}$ and $\big(\varphi(y^{(k)})\big)_{[-M_\varphi,\, -M_\varphi+|l^{(k)}_m\, l^{(k)}_n|)}$ consist of concatenations of first-return-loops from L. Now prove that these are both equal to $l^{(k)}_m\, l^{(k)}_n$ as above to establish:

$$\Big(\sigma^{-M_\varphi}\big(\varphi(y^{(k)})\big)\Big)_{[-|l^{(k)}_m\, l^{(k)}_n|,\, |l^{(k)}_m\, l^{(k)}_n|)} = \big(\varphi(y^{(k)})\big)_{[-M_\varphi-|l^{(k)}_m\, l^{(k)}_n|,\, -M_\varphi+|l^{(k)}_m\, l^{(k)}_n|)} = \\ = l^{(k)}_m\, l^{(k)}_n\, l^{(k)}_m\, l^{(k)}_n = \big(y^{(k)}\big)_{[-|l^{(k)}_m\, l^{(k)}_n|,\, |l^{(k)}_m\, l^{(k)}_n|)}$$

For k increasing $\big(\sigma^{-M_\varphi}\circ\varphi\big)(y^{(k)})$ and $y^{(k)}$ coincide on longer and longer blocks symmetric to the zero-coordinate: $\big(\big(\sigma^{-M_\varphi}\circ\varphi\big)(y^{(k)})\big)_{[-k,k]} = y^{(k)}_{[-k,k]}$. As φ is continuous the convergence $y^{(k)} \xrightarrow{k\to\infty} x$ guarantees $\varphi(y^{(k)}) \xrightarrow{k\to\infty} \varphi(x)$ and in the limit one gets the demanded result:

$$\varphi(x) = \lim_{k\to\infty}\Big(\big(\varphi\circ\sigma^{-M_\varphi}\big)\big(\sigma^{M_\varphi}(y^{(k)})\big)\Big)_{[-k,k]} = \lim_{k\to\infty}\big(\sigma^{M_\varphi}(y^{(k)})\big)_{[-k,k]} = \sigma^{M_\varphi}(x)$$

□

Combining Claims 3 and 4 φ acts on the set of points corresponding to bi-infinite, simple walks like the M_φ-th power of the shift map. □

PROPOSITION 7.2. *If a strongly connected directed graph $G = (V, E)$ has an one-element vertex-ROME $v \in V$, then every loop in G contains an edge which does not show up in any bi-infinite, simple walk.*

Proof. Suppose there is a loop $l := e_1\, e_2 \dots e_{|l|}$ ($e_i \in E$) incompatible with the statement. W.l.o.g. l is simple with $|l| \geq 2$ and $\mathfrak{i}(e_1) = \mathfrak{t}(e_{|l|}) = v$. As $e_{|l|}$ is part of some bi-infinite, simple walk there is a left-infinite, simple walk $x_- := \dots x_{-2}\, x_{-1}$ avoiding v and ending at the earliest possible vertex in l, i.e. $\mathfrak{t}(x_{-1}) = \mathfrak{t}(e_m) \neq v$ with $1 \leq m < |l|$ minimal. In the same way denote by $x_+ := x_1\, x_2 \dots$ a right-infinite, simple walk which never visits v and leaves l at the latest possible vertex $\mathfrak{i}(x_1) = \mathfrak{i}(e_n) \neq v$ with $1 < n \leq |l|$ maximal. Such x_+ exists, since e_1 shows up in some bi-infinite, simple walk in G.

If $m < n$, then one would have a bi-infinite walk $x_-\, e_{m+1} \dots e_{n-1}\, x_+$ completely avoiding the vertex-ROME v. Therefore assume $m \geq n$: By choice of m, n every bi-infinite walk containing an edge e_i ($n \leq i \leq m$) would visit v at least two times – once before e_i, once afterwards. So the edges $e_n, e_{n+1}, \dots, e_m$ can never show up in any bi-infinite, simple walk contradicting the assertion on l. □

PROPOSITION 7.3. *Let $G = (V, E)$ be a strongly connected, locally finite directed graph (with $|E| = \aleph_0$) having an one-element vertex-ROME $v \in V$. Every simple path p in G which is part of infinitely many first-return-loops at v is already contained in some bi-infinite, simple walk.*

Proof. Denote by L the set of first-return-loops at v and by (X_G, σ) the Markov shift given on G. Inductively construct a representative $x \in X_G$ of some bi-infinite, simple walk: For $b^{(0)} := p$ define the infinite set $L_{b^{(0)}} := \{p\,t\,r \mid \exists\, l \in L, r, t \text{ paths: } l = r\,p\,t\} \cup \{p\,s \mid \exists\, l \in L, r, s, t \text{ paths: } l = r\,s\,t \;\wedge\; t\,r = p\}$ of all (cyclically permuted) first-return-loops starting with p. Choose $a_n, c_n \in E$ ($n \in \mathbb{N}$) such that $b^{(n)} := a_n\, b^{(n-1)}\, c_n$ is a valid path in G and the set $L_{b^{(n)}} := \big\{l \in L_{b^{(0)}} \;\big|\; |l| \geq 2n + |p| \;\wedge\; \exists\, q \text{ path: } l = p\,q \;\wedge\; q\,p\,q \text{ contains } b^{(n)}\big\}$ remains infinite. This yields a nested sequence ${}_0[b^{(0)}] \supseteq {}_{-1}[b^{(1)}] \supseteq {}_{-2}[b^{(2)}] \supseteq \dots$ of non-empty cylinders converging to a point $x \in \bigcap_{n\in\mathbb{N}} {}_{-n}[b^{(n)}]$.

Since elements in $L_{b^{(n)}}$ have at least length $2n + |p|$ and contain exactly one edge from $F := \{e \in E \mid \mathfrak{i}(e) = v\}$, there are at most two edges from F in $q\,p\,q$ separated by a block of length $|l|-1 \geq 2n+|p|-1$. Therefore the block $b^{(n)}$ is too short to comprise more than one edge from F. As F is an edge-ROME, x has precisely one edge in F and is thus a representative of some bi-infinite, simple walk with $x_{[0,|p|-1]} = p$. □

LEMMA 7.4. *Let (X_G, σ) be an edge shift on some thinned-out graph $G=(V,E)$, $v \in V$ a vertex-ROME and L the set of first-return-loops at v. For any automorphism $\varphi \in \mathrm{Aut}(\sigma)$ there is an integer $M_\varphi \in \mathbb{Z}$ and a finite set $K_\varphi \subsetneq E$ of edges lying in the complement of all bi-infinite, simple walks in G such that $\varphi|_{\mathrm{Orb}(K_\varphi)^\complement} = \sigma^{M_\varphi}|_{\mathrm{Orb}(K_\varphi)^\complement}$.*
Moreover the condition that none of the edges in K_φ is contained in some bi-infinite, simple walk is equivalent to K_φ marking only finitely many elements in L.

Proof. Denote by $M_\varphi \in \mathbb{Z}$ the integer found in Lemma 7.1, i.e. φ acts like σ^{M_φ} on all representatives of bi-infinite, simple walks in G. To prove the main statement one has to construct a finite set $K_\varphi \subsetneq E$ such that the back-shifted automorphism $\varphi \circ \sigma^{-M_\varphi}$ induces the identity on $\mathrm{Orb}(K_\varphi)^\complement$. (The notation K_φ instead of $K_{\varphi\circ\sigma^{-M_\varphi}}$ is justified, since $\mathrm{Orb}(K_\varphi)^\complement$ is shiftinvariant and thus K_φ is instantly valid for all automorphisms $\varphi \circ \sigma^i$ $(i \in \mathbb{Z})$.)

Let $F := \{e \in E \mid \mathrm{i}(e) = v\}$ be the set of out-going edges at v and $I \in \mathbb{N}$ a global bound for the distance from any edge in $K \subsetneq E$ given as in (FP) to the nearest edge from F. Since $\bigcup_{f\in F} {}_0[f]$ is compact-open there is a common coding-length $J \geq I$ for all edges in F such that for all representatives $x \in X_G$ of bi-infinite, simple walks with $x_0 \in F$ and $y \in {}_{-J}[x_{-J} \ldots x_J]$ the zero-coordinates of the image points coincide: $\big(\varphi \circ \sigma^{-M_\varphi}(y)\big)_0 = \big(\varphi \circ \sigma^{-M_\varphi}(x)\big)_0 = x_0 \in F$. Once more the validity of (GC) allows one to choose $N \in \mathbb{N}$ such that $|l_N| > 2J$ and $\big||l_i| - |l_j|\big| > 2I$ for all $i \geq N$ and $j \neq i$.

As there exist only finitely many paths of length J $(J+1)$ ending (starting) at v, almost all elements in L start with some block from $B_+ := \big\{y_{[0,J]} \bigm| y \in \bigcup_{f\in F} {}_0[f] \ \wedge\ y$ represents some bi-infinite, simple walk$\big\}$ and end in a block from $B_- := \big\{y_{[-J,-1]} \bigm| y \in \bigcup_{f\in F} {}_0[f] \ \wedge\ y$ represents some bi-infinite, simple walk$\big\}$. To show this, define finite sets $A_+ := \big\{y_{[0,J]} \bigm| y \in \bigcup_{f\in F} {}_0[f] \ \wedge\ y_{[0,J]}$ a simple path$\big\}$ and $A_- := \big\{y_{[-J,-1]} \bigm| y \in \bigcup_{f\in F} {}_0[f] \wedge y_{[-J,-1]}$ a simple path$\big\}$. A_+ (A_-) comprises the prefixes (suffixes) of all first-return-loops at v of length greater than $J+2$. According to Proposition 7.3 any $a \in A_+$ (or $a \in A_-$) being part of infinitely many first-return-loops at v is already an element in B_+ (or B_-). Thus there is only a finite set of exceptional elements in L.

Define $L_\varphi := \{l_i \in L \mid i \geq N \ \wedge\ \exists\, u \in B_+, w \in B_-, b \text{ path: } l_i = u\,b\,w\}$. Its complement $L \setminus L_\varphi$ is finite. Using Proposition 7.2 one can build up K_φ taking from every element in $L \setminus L_\varphi$ a single edge which is not part in any bi-infinite, simple walk.

Obviously every point in $\mathrm{Orb}(K_\varphi)^\complement$ can be approximated by a convergent sequence of points being infinite concatenations of elements from L_φ. For such points $x \in {L_\varphi}^\infty$ the equality $\varphi \circ \sigma^{-M_\varphi}(x) = x$ can be established using the same arguments as in the proof of Lemma 7.1. Continuity of $\varphi \circ \sigma^{-M_\varphi}$ then shows $\varphi \circ \sigma^{-M_\varphi}|_{\mathrm{Orb}(K_\varphi)^\complement} = \mathrm{Id}_{\mathrm{Orb}(K_\varphi)^\complement}$.

Finally it remains to show the demanded equivalence for K_φ:

"$\Longrightarrow$": Suppose K_φ marks infinitely many elements in L then this is already true for some $k \in K_\varphi$ and this implies $\mathfrak{i}(k) \neq \mathfrak{t}(k)$ (self loops cannot show up in several first-return-loops). Following from Proposition 7.3 the simple path $p := k$ would be part of some bi-infinite, simple walk.

"$\Longleftarrow$": Assume $k \in K_\varphi$ shows up in a representative $x := (x_i)_{i\in\mathbb{Z}} \in X_G$ of some bi-infinite, simple walk with $\mathfrak{i}(x_0) = v$, i.e. $x_N = k$ for some $N \in \mathbb{Z}$. For $n \geq N \geq 0$ choose some minimal path $p^{(n)}$ from $\mathfrak{t}(x_n)$ back to v. This gives an infinite subset $\{x_{[0,n]}\, p^{(n)} \mid n \geq N\} \subseteq L$ of first-return-loops containing k. For $N < 0$ the infinite subset $\{q^{(n)}\, x_{[-n,-1]} \mid n \geq |N|\} \subseteq L$ with $q^{(n)}$ a shortest path from v to $\mathfrak{i}(x_{-n})$ $(n \geq |N|)$ forces the same contradiction. □

After these preparations we can easily finish the proof of Theorem 6.4. Most of the work is already done: The first statement is essentially Lemma 7.1; the existence of K_φ is shown in Lemma 7.4. What remains is the decomposition of $\mathrm{Aut}(\sigma)$:

Proof. It is easy to see that the second part of the direct sum representation $H := \{\varphi \in \mathrm{Aut}(\sigma) \mid \exists K_\varphi \subsetneq E \text{ finite, as in Lemma 7.4} \wedge \varphi|_{\mathrm{Orb}(K_\varphi)^\complement} = \mathrm{Id}_{\mathrm{Orb}(K_\varphi)^\complement}\}$ is actually a subgroup of $\mathrm{Aut}(\sigma)$: Let $\varphi,\, \phi \in H$ with corresponding finite sets $K_\varphi,\, K_\phi \subsetneq E$, then $K_{\varphi\circ\phi} := K_\varphi \cup K_\phi \subsetneq E$ is still finite and does not contain any edge from bi-infinite, simple walks. Moreover $\mathrm{Orb}(K_{\varphi\circ\phi})^\complement \subseteq \mathrm{Orb}(K_\varphi)^\complement \cap \mathrm{Orb}(K_\phi)^\complement$ such that $(\varphi\circ\phi)|_{\mathrm{Orb}(K_{\varphi\circ\phi})^\complement} = \mathrm{Id}_{\mathrm{Orb}(K_{\varphi\circ\phi})^\complement}$. Of course $\mathrm{Id}_X \in H$ (choose $K_{\mathrm{Id}_X} := \emptyset$) and $\varphi \in H$ implies $\varphi^{-1} \in H$ by means of $K_{\varphi^{-1}} := K_\varphi$.

The map $\alpha :\ \mathrm{Aut}(\sigma) \to \langle\sigma\rangle,\ \varphi \mapsto \sigma^{M_\varphi}$ with $M_\varphi \in \mathbb{Z}$ as in Lemma 7.1 is a well-defined additive homomorphism with $M_{\varphi\circ\phi} := M_\varphi + M_\phi$ for $\varphi, \phi \in \mathrm{Aut}(\sigma)$, i.e. $\alpha(\varphi\circ\phi) = \sigma^{M_{\varphi\circ\phi}} = \sigma^{M_\varphi + M_\phi} = \sigma^{M_\varphi} \circ \sigma^{M_\phi} = \alpha(\varphi)\circ\alpha(\phi)$ and $K_{\varphi\circ\phi} = K_\varphi \cup K_\phi$ as above gives $(\varphi\circ\phi)|_{\mathrm{Orb}(K_{\varphi\circ\phi})^\complement} = \sigma^{M_{\varphi\circ\phi}}|_{\mathrm{Orb}(K_{\varphi\circ\phi})^\complement}$.

The kernel of α is H (see Lemma 7.4); so there are two normal subgroups $\langle\sigma\rangle, \ker(\alpha) \trianglelefteq \mathrm{Aut}(\sigma)$ with $\langle\sigma\rangle \cdot \ker(\alpha) = \{\sigma^n \circ \varphi \mid n \in \mathbb{Z} \wedge \varphi \in \ker(\alpha)\} = \mathrm{Aut}(\sigma)$ and $\langle\sigma\rangle \cap \ker(\alpha) = \{\mathrm{Id}_X\}$ and so $\mathrm{Aut}(\sigma)$ decomposes into a direct sum.

Since thinned-out Markov shifts have countable automorphism groups (see Proposition 6.3 above and Theorem 2.4), H is countably infinite. Furthermore using Theorem 5.4 the center of $\mathrm{Aut}(\sigma)$ has to be isomorphic to $\mathbb{Z}$ for non-trivial Markov shifts. This proves H centerless. □

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