On the algebraic properties of the automorphism groups of countable state Markov shifts

MICHAEL SCHRAUDNER

Institut für Angewandte Mathematik, Universität Heidelberg
Im Neuenheimer Feld 294, 69120 Heidelberg, Germany
(e-mail: schraudner@math.uni-heidelberg.de)

Abstract. We study the algebraic properties of automorphism groups of topological, countable state Markov shifts together with the dynamics of those groups on the shiftspace itself as well as on periodic orbits and the 1-point-compactification of the shiftspace.
We present a complete solution to the cardinality-question of the automorphism group for locally compact and non locally compact, countable state Markov shifts, shed some light on its huge subgroup structure and prove the analogue of Ryan's theorem about the center of the automorphism group in the non-compact setting.
Moreover we characterize the 1-point-compactifications of locally compact, countable state Markov shifts, whose automorphism groups are countable and show that these compact dynamical systems are conjugate to synchronised systems on doubly-transitive points.

Keywords: countable state Markov shift, automorphism group, Ryan's theorem, 1-point-compactification

2000 MSC Primary: 37B10

1. Basic definitions and outline of the paper

Let $\mathcal{A}$ be a countably infinite set endowed with the discrete topology. The product space $\mathcal{A}^{\mathbb{Z}}$ (with product topology), consisting of all bi-infinite sequences of symbols from the alphabet $\mathcal{A}$, is a non-compact, totally disconnected, perfect metric space. The (left-)shift map $\sigma: \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}},\ \sigma((x_i)_{i\in\mathbb{Z}}) := (x_{i+1})_{i\in\mathbb{Z}}$ is a homeomorphism. It induces some dynamics on $\mathcal{A}^{\mathbb{Z}}$ and $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ is called the full shift on $\mathcal{A}$.
Every shift-invariant subset X of $\mathcal{A}^{\mathbb{Z}}$ endowed with the induced subspace topology together with the restriction of the shift map $\sigma = \sigma|_X$ yields a subshift (X, σ). There is a countable set of clopen cylinders ${}_n[a_0 \dots a_m] := \{(x_i)_{i\in\mathbb{Z}} \in X \mid \forall\, 0 \leq i \leq m:\ x_{n+i} = a_i\}$ $(n \in \mathbb{Z},\ m \in \mathbb{N}_0)$ generating the topology on X. Two subshifts (X_1, σ_1) and (X_2, σ_2) are (topologically) conjugate, if there is a homeomorphism $\gamma:\ X_1 \to X_2$ that commutes with the shift maps $(\sigma_2 \circ \gamma = \gamma \circ \sigma_1)$. Then (X_1, σ_1) and (X_2, σ_2) are merely two presentations of the same topological

dynamical object and we denote by $\mathrm{Pres}(X)$ the set of all presentations of the subshift (X,σ), i.e. the set of all subshifts conjugate to (X,σ).
Let $G=(V,E)$ be a directed graph with vertex set V, edge set E together with the maps $\mathfrak{i},\mathfrak{t}: E\to V$, where $\mathfrak{i}(e)$ gives the initial and $\mathfrak{t}(e)$ the terminal vertex of an edge $e\in E$. A subshift (X,σ) is called countable state Markov shift, if its set of presentations contains an edge shift (X_G,σ), with $X_G := \big\{(x_i)_{i\in\mathbb{Z}}\in E^{\mathbb{Z}} \mid \forall\, i\in\mathbb{Z}: \ \mathfrak{t}(x_i)=\mathfrak{i}(x_{i+1})\big\}$ the set of bi-infinite walks along the edges of a countably infinite directed graph G ($|E|=\aleph_0$) and σ acting on X_G. If not stated explicitly all graphs are directed, having a countably infinite set of edges. W.l.o.g. the graphs considered are assumed to be essential, i.e. the in- and out-degree at every vertex is strictly positive. (X_G,σ) is then called a graph presentation of (X,σ) and $\mathrm{Graph}(X)$ denotes the set of all graph presentations of (X,σ).
For every point $x\in X$ in a subshift (X,σ) and $m\le n\in\mathbb{Z}$ let $x_{[m,n]}$, $x_{[m,\infty)}$ and $x_{(-\infty,n]}$ respectively denote the block $x_m\,x_{m+1}\ldots x_{n-1}\,x_n$, a right- or a left-infinite ray of x. In an edge shift $x_{[m,n]}$ corresponds to a finite path of length $n-m+1$, whereas $x_{[m,\infty)}$ and $x_{(-\infty,n]}$ are equivalent to right- and left-infinite walks.
We define the language $\mathcal{B}(X)$ of a subshift (X,σ) as the disjoint union of all sets of blocks $\mathcal{B}_m(X) := \big\{x_{[0,m-1]} \bigm| x\in X\big\}\subseteq\mathcal{A}^m$ ($m\in\mathbb{N}$). $|w|$ denotes the length and w^n ($n\in\mathbb{N}_0\,\dot\cup\,\{\infty\}$) the n-times concatenation of a block $w\in\mathcal{B}(X)$.
A subshift (X,σ) is called locally compact, if X is locally compact. For countable state Markov shifts this implies the compactness of every cylinder set. An edge shift (X_G,σ) is locally compact, iff every vertex in G has finite in- and out-degree (G is locally finite).
A subshift (X,σ) is called (topologically) transitive, if X is irreducible, i.e. for every pair $u,w\in\mathcal{B}(X)$ of blocks there is a block $v\in\mathcal{B}(X)$, such that $u\,v\,w\in\mathcal{B}(X)$. An edge shift (X_G,σ) is transitive, iff G is strongly connected.
Let $\mathrm{Orb}(X) := \{\mathrm{Orb}(x)\mid x\in X\}$ the set of σ-orbits $\mathrm{Orb}(x) := \{\sigma^n(x)\mid n\in\mathbb{Z}\}\subseteq X$. Using the backward-orbit $\mathrm{Orb}^-(x) := \{\sigma^{-n}(x)\mid n\in\mathbb{N}_0\}$ and the forward-orbit $\mathrm{Orb}^+(x) := \{\sigma^n(x)\mid n\in\mathbb{N}_0\}$ we define the set of doubly-transitive points $\mathrm{DT}(X) := \big\{x\in X\mid \mathrm{Orb}^-(x),\,\mathrm{Orb}^+(x)\text{ both are dense in }X\big\}$. For transitive subshifts this set is non-empty and dense. Let $x\in\mathrm{DT}(X)$ then every block $w\in\mathcal{B}(X)$ is contained infinitely often in $x_{(-\infty,0]}$ and $x_{[0,\infty)}$.
Finally we define the set of periodic points $\mathrm{Per}(X) := \bigcup_{n\in\mathbb{N}}\mathrm{Per}_n(X) = \dot{\bigcup}_{n\in\mathbb{N}}\mathrm{Per}^0_n(X)$ under the action of σ, where $\mathrm{Per}_n(X)$ denotes the set of points of period n and $\mathrm{Per}^0_n(X)$ the set of points of least period n. For transitive, countable state subshifts $\mathrm{Per}(X)$ is a countable dense subset in X.
For further notions and background information on subshifts we refer to the monographs on symbolic dynamics by D. Lind and B. Marcus [**LM**] and by B. Kitchens [**Kit**].

Now we recall the fundamental definition of this paper: Let (X,σ) be some subshift. A map $\varphi: X\to X$ is called an automorphism (of σ), if φ is a self-conjugacy, i.e. a shiftcommuting homeomorphism from X onto itself. Obviously the set of automorphisms forms a group $\mathrm{Aut}(\sigma)$ under composition. It is an invariant of topological conjugacy reflecting the inner structure and symmetries of the subshift. For subshifts of finite type (SFTs) there is an extensive and profound theory dealing with automorphisms (see e.g. [**BK1**], [**BK2**], [**BLR**], [**FieU1**], [**FieU2**], [**Hed**], [**KR1**], [**KR2**], [**KRW1**] and [**KRW2**]) and leading to very deep and strong results concerning the conjugacy problem, the FOG-conjecture or the LIFT-hypothesis. The automorphism group of any nontrivial SFT is a countably infinite, residually finite group (therefore it cannot contain any infinite simple or any nontrivial divisible subgroup) with center isomorphic to $\mathbb{Z}$. It is discrete with respect to the compact-open topology, does not contain any finitely generated subgroups with unsolvable word problem, but admits embeddings of a great variety of other groups (see section 4).

The automorphism groups of coded systems have been studied in [**FF2**] with quite different results (they are much smaller and can be stipulated explicitely; their center can be isomorphic to a wide range of abstract groups), whereas to our knowledge there are yet no published results on automorphisms of countable state Markov shifts.

Trying to fill part of this gap, the present paper contains results from the authors Ph.D. thesis [**Sch**]. In section 2 we determine the cardinality of $\mathrm{Aut}(\sigma)$ for locally compact and non locally compact, countable state Markov shifts and give several equivalent criteria for $\mathrm{Aut}(\sigma)$ being countable. In section 3 we study the 1-point-compactifications of locally compact, countable state Markov shifts with $\mathrm{Aut}(\sigma)$ countable. Those compact dynamical systems need no longer be expansive, that is in general they aren't conjugate to any subshift. Instead the property $\mathrm{Aut}(\sigma)$ countable is equivalent to expansiveness being restricted to doubly-transitive points. Furthermore this implies the existence of an almost invertible 1-block-factor-map from the compactification onto some synchronised system. Section 4 contains some results on the subgroup structure of $\mathrm{Aut}(\sigma)$. Like in the SFT-case we can realize lots of abstract groups via marker constructions. The gradual fading of compactness (**(FMDP)**, locally compact, non locally compact) shows up in a decrease of algebraic restrictions and an increase of possible subgroups. This makes it very difficult to describe $\mathrm{Aut}(\sigma)$ as an abstract group. Stimulated by the well-known result of J. Ryan [**Rya1**],[**Rya2**] on the center of $\mathrm{Aut}(\sigma)$ for SFTs, we are able to reprove this theorem for non-compact Markov shifts in section 5. Therefore $\mathrm{Aut}(\sigma)$ is again highly non-abelian (in contrast to the coded-systems-case) and the periodic-orbit representation is faithful on ${}^{\mathrm{Aut}(\sigma)}/_{\langle\sigma\rangle}$.

2. The cardinality of $\mathrm{Aut}(\sigma)$

Let $\mathcal{S}_{\mathbb{N}}$ be the set of all bijective mappings from $\mathbb{N}$ (or generally any countably infinite set) onto itself. We call $\mathcal{S}_{\mathbb{N}}$ the full permutation group (on a countable set). Its cardinality is $2^{\aleph_0}$. By $\mathcal{S}_{\mathbb{N},\mathfrak{f}}$ we denote the subgroup of finite permutations, i.e. the set of all bijective mappings from $\mathbb{N}$ onto itself that fix all but finitely many elements. The cardinality of $\mathcal{S}_{\mathbb{N},\mathfrak{f}}$ is $\aleph_0$.

Proposition 2.1. *The automorphism group of every transitive, countable state Markov shift is isomorphic to a subgroup of $\mathcal{S}_{\mathbb{N}}$ and therefore has cardinality at most $2^{\aleph_0}$.*

Proof: Since the Markov shift (X,σ) is transitive, the countable set of periodic points $\mathrm{Per}(X)$ is dense in X and every automorphism $\varphi \in \mathrm{Aut}(\sigma)$ is uniquely determined by its action on $\mathrm{Per}(X)$. Therefore $\mathrm{Aut}(\sigma) \leq \mathcal{S}_{\mathrm{Per}(X)} \cong \mathcal{S}_{\mathbb{N}}$. □

Now we can state the cardinality-result for non locally compact Markov shifts:

Theorem 2.2. *Every transitive, non locally compact, countable state Markov shift has an automorphism group of cardinality $2^{\aleph_0}$.*

Proof: Let $G = (V, E)$ be a graph presentation for the non locally compact Markov shift (X, σ). W.l.o.g. we may assume that there is a vertex $v \in V$ with infinite out-degree (the symmetric situation of a vertex with infinite in-degree can be treated via time-reversal, i.e. carrying out the following construction for the transposed graph).
Let $\{e_j \mid j \in \mathbb{N}\} \subseteq E$ be the set of edges starting at v. For every $j \notin 3\mathbb{N}$ choose a shortest path p_j from $\mathfrak{t}(e_j)$ back to v (G is strongly connected); p_j is empty, if $\mathfrak{t}(e_j) = v$. This gives an infinite set of distinct loops $l_j := e_j\, p_j$ at the vertex v. Use the edges e_j ($j \in 3\mathbb{N}$) as markers to define maps $\phi_i : X \to X$ ($i \in \mathbb{N}$) that interchange the blocks $l_{3i-2}\, l_{3i-1}\, e_{3i}$ and $l_{3i-1}\, l_{3i-2}\, e_{3i}$ in every point $x \in X$ and take no further action.
By construction no path p_j ($j \notin 3\mathbb{N}$) can contain an edge e_i ($i \in \mathbb{N}$). This guarantees that no loop l_j ($j \notin 3\mathbb{N}$) contains any edge e_{3i} and no two loops can overlap partially. Therefore every ϕ_i is well-defined. ϕ_i is an involutorial sliding-block-code with coding length $2\,|l_{3i-2}\, l_{3i-1}| + 1$. So we have constructed a countable set of distinct automorphisms $\{\phi_i \mid i \in \mathbb{N}\} \subseteq \mathrm{Aut}(\sigma)$.
Next consider infinite products of the maps ϕ_i and show that for every 0/1-sequence $(a_k)_{k\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ there is a well-defined automorphism $\varphi_{(a_k)} := \prod_{i\in\mathbb{N}} {\phi_i}^{a_i}$:
Distinct automorphisms ϕ_i act on disjoint blocks ending with the symbol e_{3i}. Furthermore the $\{e_{3i} \mid i \in \mathbb{N}\}$-skeleton, i.e. the coordinates at which a symbol e_{3i} appears remain invariant under any composition of ϕ_i s. The ϕ_i commute with each other and the infinite product $\varphi_{(a_k)}$ is defined independently of the order of composition. We get $(\varphi_{(a_k)})^2 = (\prod_{i\in\mathbb{N}} {\phi_i}^{a_i})^2 = \prod_{i\in\mathbb{N}} {\phi_i}^{2a_i} = \mathrm{Id}_X$ and $\varphi_{(a_k)}(X) \subseteq X$, so $\varphi_{(a_k)}$ is a well-defined order 2 bijection from X onto X and obviously $\varphi_{(a_k)}$ commutes with the shift map.
To show that $\varphi_{(a_k)}$ (and ${\varphi_{(a_k)}}^{-1} = \varphi_{(a_k)}$) is continuous, it suffices to show that the zero-coordinate of the image is prescribed by a finite block of the preimage:
Fix $x \in X$. The symbol x_0 is unchanged unless it is part of some block $l_{3i-2}\, l_{3i-1}\, e_{3i}$ or $l_{3i-1}\, l_{3i-2}\, e_{3i}$. Let n be the length of a shortest path from $\mathfrak{t}(x_0)$ to the vertex v. Whenever $x_{n+1} \notin \{e_i \mid i \in \mathbb{N}\}$ we have $\big(\varphi_{(a_k)}(x)\big)_0 = x_0$. In this case the block $x_{[0,n+1]}$ decides about the zero-coordinate of the image. If $x_{n+1} = e_j$ for some $j \in \mathbb{N}$ the only automorphism in the product that can act on the zero-coordinate is ϕ_i with $i := \lceil \frac{j}{3} \rceil$. We have $\big(\varphi_{(a_k)}(x)\big)_0 = \big({\phi_i}^{a_i}(x)\big)_0$. Since ϕ_i is sliding-block, $\big(\varphi_{(a_k)}(x)\big)_0$ is determined by the knowledge of a finite block of x. As $\varphi_{(a_k)}$ commutes with σ, this proves continuity.
Two distinct 0/1-sequences $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ define distinct automorphisms $\varphi_{(a_k)}, \varphi_{(b_k)}$. For $i \in \mathbb{N}$ such that $a_i \neq b_i$, the point $x := (l_{3i-2}\, l_{3i-1}\, e_{3i}\, p_{3i})^\infty \in X$ (p_{3i} a shortest path from $\mathfrak{t}(e_{3i})$ back to v) has different images under $\varphi_{(a_k)}$ and $\varphi_{(b_k)}$. Therefore we have constructed a subgroup $\big\{\varphi_{(a_k)} \mid (a_k)_{k\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}\big\} \leq \mathrm{Aut}(\sigma)$ of cardinality $2^{\aleph_0}$. □

We remark that though all ϕ_i in the proof of theorem 2.2 are sliding-block-codes, i.e. uniformly continuous maps, the infinite products $\varphi_{(a_k)}$ needn't have bounded coding length and are in general merely continuous.

To answer the cardinality-question for locally compact, countable state Markov shifts we need the notion of a doublepath in a directed graph:
A pair of two distinct paths p, q of equal length ($|p| = |q|$), connecting the same initial with the same terminal vertex is called a doublepath and is denoted $[p;q]$. By definition we have $[p;q] = [q;p]$ and in slight abuse of notation $\mathfrak{i}(p) = \mathfrak{i}(q)$ and $\mathfrak{t}(p) = \mathfrak{t}(q)$.
Two doublepaths $[p_1;q_1]$ and $[p_2;q_2]$ are edge-disjoint, if the union of all edges in p_1 and q_1 is disjoint from all edges in p_2 union q_2.

A strongly connected, directed graph has the property **(FMDP)**, if it contains at most **F**initely **M**any pairwise edge-disjoint **D**ouble**P**aths.

THEOREM 2.3. *Let (X,σ) be a transitive, locally compact, countable state Markov shift.* $\mathrm{Aut}(\sigma)$ *has cardinality $\aleph_0$, iff any (every) graph presentation of (X,σ) has* **(FMDP)**. *Otherwise* $\mathrm{Aut}(\sigma)$ *has cardinality $2^{\aleph_0}$.*

The proof of theorem 2.3 is given in three steps:

LEMMA 2.4. *Let (X_G,σ) be any graph presentation of a transitive, locally compact, countable state Markov shift on some directed graph G containing infinitely many, pairwise edge-disjoint doublepaths. Then* $\mathrm{Aut}(\sigma)$ *has cardinality $2^{\aleph_0}$.*

PROOF: Since X_G is irreducible and locally compact, G has to be strongly connected and locally finite. Let $P := \{[p_i;q_i] \mid i \in \mathbb{N}\}$ be an infinite set of pairwise edge-disjoint doublepaths in G. For every $[p_i;q_i]$ choose a marker edge e_i starting at $\mathfrak{t}(p_i) = \mathfrak{t}(q_i)$ that is not contained in this doublepath. This is possible, since both paths p_i, q_i may be extended by the same finite set of edges already contained in $[p_i;q_i]$ until they end at a vertex, at which an edge not contained in $[p_i;q_i]$ starts. Take such an edge as marker and use the enlarged doublepath in place of $[p_i;q_i]$.
Inductively we construct an infinite subset $Q \subseteq P$ of doublepaths (with adjoint markers) such that all marker edges are distinct and no one does occur in any of the doublepaths in Q: Let $Q := \emptyset$. Choose $[p;q] \in P$; define $Q := Q \,\dot{\cup}\, \{[p;q]\}$. Due to the local finiteness of G there are at most finitely many elements in the set P whose markers are part of $[p;q]$. After removing this finite subset, the element $[p;q]$ itself as well as the doublepath (if there is one) containing the marker of $[p;q]$ from P, we are left with a still infinite set. Choosing one of the remaining doublepaths we iterate this procedure to build up an infinite subset Q as desired. For simplicity of notation renumber the elements in Q to get $Q = \{[p_i;q_i] \mid i \in \mathbb{N}\}$.
For every 0/1-sequence $(a_k)_{k\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ define a map $\varphi_{(a_k)} : X_G \to X_G$ that interchanges every block $p_i\, e_i$ and $q_i\, e_i$ in a point in X_G, iff $a_i = 1$. Caused by edge-disjointness of the doublepaths $[p_i;q_i] \in Q$ and the use of the distinct markers e_i, being edge-disjoint from all elements in Q, no partial overlaps are possible and $\varphi_{(a_k)}$ is well-defined. $\varphi_{(a_k)}$ commutes with σ by construction. Furthermore $\varphi_{(a_k)}(X_G) \subseteq X_G$ and ${\varphi_{(a_k)}}^2 = \mathrm{Id}_{X_G}$, that is $\varphi_{(a_k)} = {\varphi_{(a_k)}}^{-1}$ is bijective.

Continuity of $\varphi_{(a_k)}$ is shown as in the proof of theorem 2.2. The zero-coordinate of $x \in X_G$ is unchanged unless x_0 is part of a by definition of Q uniquely determined doublepath $[p_j; q_j] \in Q$. Looking at the finite block $x_{[1-|p_j|,|p_j|]}$ one can decide about $\big(\varphi_{(a_k)}(x)\big)_0$: Suppose $x_{[m,m+|p_j|]} = p_j\, e_j$ for some $1-|p_j| \leq m \leq 0$ and $a_j = 1$, then the zero-coordinate of the image has to be the $(1-m)$-th symbol of the block q_j. Analogously for $x_{[m,m+|p_j|]} = q_j\, e_j$. In all other cases $\big(\varphi_{(a_k)}(x)\big)_0 = x_0$. Therefore $\varphi_{(a_k)}$ is a shiftcommuting homeomorphism.
Obviously distinct sequences $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ give rise to distinct maps $\varphi_{(a_k)} \neq \varphi_{(b_k)}$, because for $i \in \mathbb{N}$ such that $a_i \neq b_i$ the images $\varphi_{(a_k)}(x)$ and $\varphi_{(b_k)}(x)$ of a point $x \in X_G$ with $x_{[0,|p_i|]} = p_i\, e_i$ differ. This shows the existence of a subgroup $\big\{\varphi_{(a_k)} \mid (a_k)_{k\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}\big\} \leq \mathrm{Aut}(\sigma)$ of cardinality $2^{\aleph_0}$. □

For the next lemma we need the notion of the F-skeleton of a bi-infinite sequence: Let $F \subseteq \mathcal{A}$ be a subset of some alphabet $\mathcal{A}$. The F-skeleton of a point $x \in \mathcal{A}^{\mathbb{Z}}$ is the partial map $\kappa_x : \mathbb{Z} \to F$, $\kappa_x(i) := \begin{cases} x_i & \text{if } x_i \in F \\ \uparrow & \text{otherwise} \end{cases}$ ($\uparrow$ signals an undefined value of κ_x).

Lemma 2.5. *A locally finite, strongly connected graph $G = (V,E)$ has property* **(FMDP)**, *iff there is a finite set $F \subseteq E$ of edges, such that every doubly-transitive walk along the edges of G is uniquely determined by its F-skeleton.*

Proof: W.l.o.g. we may assume $|E| = \aleph_0$, since otherwise $F := E$ is a good choice to prove the statement.
"$\Longleftarrow$": Suppose G does not have **(FMDP)**, then for every finite set $F \subsetneq E$ there is a doublepath $[p;q]$ that does not contain an edge from F (in fact there are infinitely many). The path p occurs infinitely often in every doubly-transitive walk. Exchanging one such block p by the block q gives another doubly-transitive walk, that obviously has the same F-skeleton.
"$\Longrightarrow$": Assume that $P := \{[p_n; q_n] \mid 1 \leq n \leq N\}$ is a maximal, finite set of pairwise edge-disjoint doublepaths in G (having **(FMDP)**). Let $F \subsetneq E$ be the union of all edges that show up in elements of P. F is a finite set. Suppose there are two doubly-transitive walks $x, y \in \mathrm{DT}(X_G)$ with the same F-skeleton. There are coordinates $i \leq j \in \mathbb{Z}$ such that $x_{i-1} = y_{i-1}$, $x_{j+1} = y_{j+1} \in F$, x_k, $y_k \notin F$ for all $i \leq k \leq j$ and $x_{[i,j]} \neq y_{[i,j]}$. This implies the existence of a doublepath $[x_{[i,j]}; y_{[i,j]}]$ of length $j-i+1$ connecting $\mathfrak{t}(x_{i-1}) = \mathfrak{t}(y_{i-1})$ with $\mathfrak{i}(x_{j+1}) = \mathfrak{i}(y_{j+1})$, which is edge-disjoint to all elements in P. This contradicts the maximality of P. □

This equivalent reformulation of the property **(FMDP)** is enough to finish the proof of theorem 2.3:

Lemma 2.6. *Let the transitive, locally compact, countable state Markov shift (X,σ) be presented on some directed graph $G = (V,E)$. Suppose there is a finite set $F \subsetneq E$ of edges such that every doubly-transitive point in X is uniquely determined by its F-skeleton, then* $\mathrm{Aut}(\sigma)$ *is countably infinite.*

Proof: Again G has to be a stongly connected, locally finite graph with $|E| = \aleph_0$. As all powers of σ are distinct automorphisms, $\mathrm{Aut}(\sigma)$ has at least cardinality $\aleph_0$. Since $\mathrm{DT}(X)$ forms a dense subset in X, every automorphism $\varphi \in \mathrm{Aut}(\sigma)$ is

uniquely determined by its action on the doubly-transitive points. It suffices to show that there are at most countably many restrictions $\varphi|_{\mathrm{DT}(X)}$ possible.
Let $F \subsetneq E$ be a finite set as stated in the lemma. X is locally compact, so every zero-cylinder ${}_0[e]$ $(e \in E)$ is compact-open. The preimages $\varphi^{-1}({}_0[e])$ are also compact-open and therefore can be covered by a finite set of cylinders. Select such a cover with (minimal) cardinality $m_f \in \mathbb{N}$ for all $f \in F$:

$$\varphi^{-1}\big({}_0[f]\big) = \bigcup_{i=1}^{m_f} {}_{n_{f,i}}[b_{f,i}] \quad \text{with } b_{f,i} \in \mathcal{B}(X) \text{ and } n_{f,i} \in \mathbb{Z}$$

As φ commutes with σ one gets:

$$\varphi\big({}_{n_{f,i}+k}[b_{f,i}]\big) \subseteq {}_k[f] = \bigcup_{j=1}^{m_f} \varphi\big({}_{n_{f,j}+k}[b_{f,j}]\big) \quad \forall\, 1 \le i \le m_f,\ k \in \mathbb{Z}$$

Knowing these finite preimage cylindersets $\big\{{}_{n_{f,i}}[b_{f,i}] \mid 1 \le i \le m_f\big\}$ for all $f \in F$ is equivalent to knowing the whole F-skeleton of the image of every point in X under φ. Since an automorphism maps $\mathrm{DT}(X)$ onto itself, this knowledge fixes $\varphi|_{\mathrm{DT}(X)}$.
Let M be the set of all mappings $\mu : F \to \{C \subsetneq \mathcal{C}(X) \mid C \text{ finite}\},\ f \mapsto \big\{{}_{n_{f,i}}[b_{f,i}] \mid 1 \le i \le m_f\big\}$ where $\mathcal{C}(X)$ denotes the countable set of all cylinders of X. Obviously $\{C \subsetneq \mathcal{C}(X) \mid C \text{ finite}\}$ is countable and so is M. Now a mapping $\mu \in M$ induces at most one automorphism, so there is an injection from $\mathrm{Aut}(\sigma)$ into M, proving $\mathrm{Aut}(\sigma)$ countable. □

Since the automorphism group is an invariant of topological conjugacy, its cardinality is independent of the subshift presentation one has chosen. In particular this shows the conjugacy-invariance of the property **(FMDP)** as claimed in theorem 2.3: Either every or no graph presentation of a given transitive, locally compact, countable state Markov shift has **(FMDP)**.

As a direct consequence of lemmata 2.4 and 2.6 we get a result about the compact-open topology on $\mathrm{Aut}(\sigma)$. This topology is build up from subbasis sets of the form $S(C,U) := \{\varphi \in \mathrm{Aut}(\sigma) \mid \varphi(C) \subseteq U\}$, where $C \subseteq X$ is compact and $U \subseteq X$ is open.
For SFTs the compact-open topology on $\mathrm{Aut}(\sigma)$ is known to be discrete (see [**Kit**], Observation 3.1.2), whereas for countable state Markov shifts this need not be true:

Corollary 2.7. *Let (X,σ) be a transitive, locally compact, countable state Markov shift. The compact-open topology on* $\mathrm{Aut}(\sigma)$ *is discrete, iff* $\mathrm{Aut}(\sigma)$ *has cardinality* $\aleph_0$.

Proof: "$\Longleftarrow$": Using the notation of lemma 2.6, every automorphism $\varphi \in \mathrm{Aut}(\sigma)$ is uniquely determined by fixing the finite sets of cylinders $\big\{{}_{n_{f,i}}[b_{f,i}] \,\big|\, 1 \le i \le m_f\big\}$ for all $f \in F$. Since φ induces a bijection on the periodic points, it is not possible to have another automorphism, whose preimage cylindersets contain that of φ for all $f \in F$. Therefore the singleton $\{\varphi\}$ can be expressed as a finite intersection of subbasis sets:

$$\bigcap_{f\in F} S\Big(\bigcup_{i=1}^{m_f} {}_{n_{f,i}}[b_{f,i}], {}_0[f]\Big) = \bigcap_{f\in F} \Big\{\phi \in \mathrm{Aut}(\sigma) \;\Big|\; \phi\big(\bigcup_{i=1}^{m_f} {}_{n_{f,i}}[b_{f,i}]\big) \subseteq {}_0[f]\Big\} = \{\varphi\}$$

"$\Longrightarrow$": Suppose $\mathrm{Aut}(\sigma)$ is not countable. Then every graph presentation contains an infinite set of pairwise edge-disjoint doublepaths. Finite intersections of subbasis sets fix the action of an automorphism merely on a finite set of these doublepaths. To single out an automorphism $\varphi_{(a_k)}$ as defined in lemma 2.4 one would need to fix the action on all doublepaths, but this is only possible via infinite intersections of subbasis sets. □

So the property **(FMDP)** not only governs the cardinality but also the topological structure of $\mathrm{Aut}(\sigma)$ for locally compact, countable state Markov shifts. In section 4 we will see that **(FMDP)** in addition has a heavy impact on the subgroup structure of the automorphism group.

3. The 1-point-compactifications of locally compact Markov shifts with $\mathrm{Aut}(\sigma)$ countable

As for any locally compact topological space one defines the 1-point-compactification (X_0, σ_0) of a transitive, locally compact, countable state Markov shift (X, σ) where $X_0 := X \dot{\cup} \{\infty\}$ denotes the Alexandroff-compactification of X and the homeomorphism $\sigma_0 : X_0 \to X_0$ is the canonical extension of the shift map: $\sigma_0|_X := \sigma$ and $\sigma_0(\infty) = \infty$.
We remark that in general σ_0 is no longer expansive. As a consequence although X_0 is still a zero-dimensional topological space, the compactification (X_0, σ_0) is a compact-metric dynamical system that need not be (conjugate to) any subshift.
D. Fiebig [**FieD**] has shown that σ_0 is expansive, i.e. (X_0, σ_0) is a subshift, if and only if any (every) graph presentation of (X, σ) on a locally finite, strongly connected graph $G = (V, E)$ contains a finite set $F \subsetneq E$ of edges such that:

(1) Every bi-infinite walk along the edges of G contains an edge from F.

(2) For any pair of edges $c, d \in E$ and $n \in \mathbb{N}$ there is at most one path $p := e_1 e_2 \dots e_n$ such that $\mathfrak{i}(e_1) = \mathfrak{t}(c)$, $\mathfrak{t}(e_n) = \mathfrak{i}(d)$ and $e_i \notin F$ for all $1 \le i \le n$.

(3) For every edge $e_0 \in E$ there is at most one right-infinite ray $r := e_0 e_1 e_2 \dots$ with $e_i \notin F$ for all $i \ge 1$ and at most one left-infinite ray $l := \dots e_{-2} e_{-1} e_0$ with $e_i \notin F$ for all $i \le -1$.

Lets have a look at property **(2)** first:

Proposition 3.1. *A strongly connected, locally finite graph G has property* **(2)***, iff it has* **(FMDP)***.*

Proof: "$\Longrightarrow$": Suppose there are infinitely many pairwise edge-disjoint doublepaths in G. To fulfill **(2)** the set F has to contain at least one edge from every doublepath. This contradicts the finiteness of F.
"$\Longleftarrow$": Let P be a maximal finite set of pairwise edge-disjoint doublepaths in G. Define $F := \{e \in E \mid \exists\, [p; q] \in P : \ e \in p \ \vee \ e \in q\} \subsetneq E$ to be the union of all edges that occur in elements of P. Since P was maximal, every doublepath in G contains an edge from the finite set F, i.e. F satisfies property **(2)**. □

Putting together theorem 2.3, proposition 3.1 and the result by D. Fiebig ([**FieD**], lemma 4.1) we get:

Corollary 3.2. *The automorphism group of every transitive, locally compact, countable state Markov shift having an expansive 1-point-compactification is countably infinite.*

Remark: There is another, more direct proof for this corollary, that does not refer to a graph presentation, but shows that even the set of endomorphisms $\mathrm{End}(\sigma)$ (continuous, shiftcommuting maps from X to itself) is countable:
Under the assumptions of corollary 3.2 the 1-point-compactification (X_0, σ_0) is a compact subshift. Due to Curtis-Hedlund-Lyndon [**Hed**] every endomorphism $\phi_0 : X_0 \to X_0$ is a sliding-block-code. Therefore $\mathrm{End}(\sigma_0)$ is at most countable. In addition there is a canonical injection $\varepsilon : \mathrm{End}(\sigma) \to \mathrm{End}(\sigma_0)$, $\phi \mapsto \phi_0$ such that $\phi_0|_X = \phi$ and $\phi_0(\infty) = \infty$, proving $\mathrm{End}(\sigma)$ countable.

In a little digression we show that the three properties in D. Fiebig's characterisation of expansiveness are not independent from each other. In fact **(3)** already implies **(1)**, forcing the equivalence between σ_0 being expansive and properties **(2)** and **(3)** alone.

Lemma 3.3. *Every strongly connected, locally finite graph containing a finite set of edges that fulfill* **(3)***, automatically satisfies property* **(1)***.*

Proof: Let $G = (V, E)$ be a directed graph as desired and $F \subsetneq E$ be a finite set satisfying **(3)**; X_G denotes the set of bi-infinite walks along the edges of G.
Suppose property **(1)** could not be fulfilled, i.e. there is an infinite set $W := \{w^{(i)} \in X_G \mid i \in \mathbb{N}\}$ of bi-infinite walks, such that no finite set of edges is enough to mark all elements in W. W.l.o.g. assume that no two elements $w^{(i)}$, $w^{(j)} \in W$ differ only by some translation ($\forall\, k \in \mathbb{Z} :\ \sigma^k(w^{(i)}) \neq w^{(j)}$) and no $w^{(i)}$ contains an edge from F.
To show that the elements of W are even pairwise edge-disjoint, suppose there is an edge $e \in w^{(i)}$ being also part of $w^{(j)}$ ($i \neq j \in \mathbb{N}$). Then $w^{(i)}$ and $w^{(j)}$ branch somewhere before (or after) e. This would give two distinct left-(right-)infinite walks ending (starting) at e that do not contain any edge from F. This clearly contradicts the assumption on F satisfying **(3)**, so the elements of W are pairwise edge-disjoint.
Let $I := \{\mathfrak{i}(f) \mid f \in F\} \subseteq V$ be the finite set of initial vertices of all edges in F. For every $i \in \mathbb{N}$ choose an edge e_i in $w^{(i)}$ and a shortest path p_i connecting $\mathfrak{t}(e_i)$ with one of the vertices in I. By construction these paths do not contain any edge from F. Since I is finite and W is infinite, there is a vertex $v \in I$ at which two (in fact infinitely many) paths p_i, p_j end. Now the paths $e_i\, p_i$ and $e_j\, p_j$ are distinct ($e_i \neq e_j$), end at the same vertex v and can be extended to left-infinite walks that do not contain any edge from F by attaching the left-infinite rays of $w^{(i)}$ and $w^{(j)}$ ending in $\mathfrak{i}(e_i)$, $\mathfrak{i}(e_j)$ respectively. Again this contradicts property **(3)**. □

Now we come back to the main purpose of this section, which is to find a fundamental description of the graph-property **(FMDP)** in a priori conjugacy-invariant, purely dynamical terms. This finally results in a presentation-independent characterisation of locally compact, countable state Markov shifts with $\mathrm{Aut}(\sigma)$ countable.

To achieve this we recall the definition of the Gurevich metric: Let (X_0, σ_0) be the 1-point-compactification of a locally compact subshift (X, σ). There is an unique metric $d_0 : X_0 \times X_0 \to \mathbb{R}^+$ which is consistent with the topology induced on X_0 by compactification of the topological space X. The (up to uniform equivalence) unique restriction $d := d_0|_X$ is called Gurevich metric. If a locally compact, countable state Markov shift (X, σ) is given in some graph presentation on $G = (V, E)$ there is an explicit formula for the Gurevich metric (see e.g. [**FF1**], page 627):

$$\forall\, x, y \in X : \quad d(x, y) := \sum_{n\in\mathbb{Z}} 2^{-|n|}\, |h(x_n) - h(y_n)|$$

where $h : E \to \{m^{-1} \mid m \in \mathbb{N}\}$ denotes any injective mapping from the edge set into the unit-fractions.

We have seen that the 1-point-compactifications of locally compact, countable state Markov shifts with $\mathrm{Aut}(\sigma)$ countable need not be subshifts. σ_0 is expansive with respect to the Gurevich metric, iff in addition to property **(FMDP)** the Markov shift has also property **(3)**. The following theorem exposes what can be said about the 1-point-compactifications in the absence of **(3)**:

THEOREM 3.4. *For transitive, locally compact, countable state Markov shifts* (X, σ) *property* **(FMDP)** *is equivalent to* σ_0 *being expansive (with respect to the Gurevich metric) on the doubly-transitive points, i.e. there is an expansivity constant* $c > 0$ *such that the otherwise uncountable set of c-shadowing points* $T_c(x) := \{y \in X \mid \forall\, n \in \mathbb{Z} : d(\sigma^n(x), \sigma^n(y)) \le c\}$ *is an one-element set for all* $x \in \mathrm{DT}(X)$. *In other words:* $\forall\, x \in \mathrm{DT}(X),\, y \in X : x \ne y \Rightarrow \exists\, n \in \mathbb{Z} : d(\sigma^n(x), \sigma^n(y)) > c$.

PROOF: "$\Longrightarrow$": Assume $G = (V, E)$ is a graph presentation for the Markov shift (X, σ) having **(FMDP)**. Following from lemma 2.5, there is a finite set of edges $F \subsetneq E$ uniquely determining every doubly-transitive point in X via its F-skeleton. For a given injective mapping $h : E \to \{m^{-1} \mid m \in \mathbb{N}\}$ inducing the Gurevich metric, one defines $c := \frac{1}{2} \min_{f\in F}\{\frac{1}{m} - \frac{1}{m+1} \bigm| m = h(f)^{-1}\}$. Since F is finite, $c > 0$. For $x, y \in X$, $x_0 \in F$ and $x_0 \ne y_0$ we have the estimate:

$$d(x, y) \ge |h(x_0) - h(y_0)| \ge \frac{1}{m} - \frac{1}{m+1} \ge 2c > c \quad \text{with } m := h(x_0)^{-1}$$

To obtain $d(\sigma^n(x), \sigma^n(y)) \le c$ for all $n \in \mathbb{Z}$, the F-skeleton of x and y have to agree. So for $x \in \mathrm{DT}(X)$ this implies $x = y$ and therefore $T_c(x) = \{x\}$.
"$\Longleftarrow$": Now assume $G = (V, E)$ contains infinitely many pairwise edge-disjoint doublepaths. For every $c > 0$ there exists a doublepath $[p; q]$ such that for all edges $e \in E$ contained in $[p; q]$ one has $h(e) \le \frac{c}{3}$ ($h : E \to \{m^{-1} \mid m \in \mathbb{N}\}$ as above). Every $x \in \mathrm{DT}(X)$ contains the block p infinitely often. Substituting any subset of these with q gives uncountably many distinct points $y \in X$. The following estimate shows that all of these shadow x in a distance $\le c$:

$$c \ge 3 \max\{h(e) \bigm| e \in [p; q]\} \ge \sum_{j\in\mathbb{Z}} 2^{-|j|} \max\{|h(x_i) - h(y_i)| \bigm| i \in \mathbb{Z}\} \ge$$

$$\ge \sum_{j\in\mathbb{Z}} 2^{-|j|} \left|h\big((\sigma^n(x))_j\big) - h\big((\sigma^n(y))_j\big)\right| = d(\sigma^n(x), \sigma^n(y)) \quad \forall\, n \in \mathbb{Z}$$

So $T_c(x)$ is uncountable and c cannot be an expansivity constant. □

Theorem 3.4 characterizes the dynamical systems that show up as 1-point-compactifications of transitive, locally compact, countable state Markov shifts (X,σ) with $\mathrm{Aut}(\sigma)$ countable as transitive, zero dimensional, compact-metric topological spaces equipped with a homeomorphism acting at least expansive on doubly-transitive points.
If (X,σ) additionally fulfills property **(3)**, every point is determined by its F-skeleton (for some $F \subsetneq E$ finite) and the homeomorphism is (fully) expansive with respect to the Gurevich metric. Another result by D. Fiebig ([**FieD**], lemma 4.5) shows that in this case the 1-point-compactification is already (conjugate to) a synchronised system with at most one point not containing a synchronising block, i.e. $\mathrm{SYN}(X_0) \supseteq X_0 \setminus \{\infty\}$. Property **(FMDP)** alone still implies almost-conjugacy to a synchronised system:

Proposition 3.5. *Let (X,σ) be a transitive, locally compact, countable state Markov shift with $\mathrm{Aut}(\sigma)$ countable. There is an almost-invertible 1-block-factor-code $\kappa: (X_0,\sigma_0) \to (Y,\sigma_Y)$ from the 1-point-compactification onto a synchronised system with $\mathrm{SYN}(Y) \supseteq Y \setminus \kappa(\infty)$, that is $\kappa|_{\mathrm{DT}(X_0)} : (\mathrm{DT}(X_0), \sigma_0|_{\mathrm{DT}(X_0)}) \to (\mathrm{DT}(Y), \sigma_Y|_{\mathrm{DT}(Y)})$ is a topological conjugacy on the doubly-transitive points.*

Proof: Let $G = (V,E)$ be a graph presentation for (X,σ) and $F \subsetneq E$ a finite set of edges determining every doubly-transitive point via its F-skeleton. Define $A := F \,\dot\cup\, \{\uparrow\}$. The skeleton map $\kappa:\ X_0 \to A^{\mathbb{Z}}:\ \big(\kappa(x)\big)_i := \begin{cases} x_i & \text{if } x_i \in F \\ \uparrow & \text{otherwise} \end{cases}$ $\forall x \in X,\ i \in \mathbb{Z}$ and $\kappa(\infty) := \uparrow^{\infty}$ is a 1-block-map, thus continuous and shiftcommuting. As X_0 is compact, so is $Y := \kappa(X_0)$; $(Y,\sigma_Y) \subseteq (A^{\mathbb{Z}},\sigma)$ is a compact subshift.
Every symbol $f \in F$ is a synchronising block in (Y,σ_Y): Let $\tilde{x},\tilde{y} \in Y$ with $\tilde{x}_0 = f = \tilde{y}_0$. Since X is given in graph presentation, all preimages $x \in \kappa^{-1}(\tilde{x}) \subseteq X$, $y \in \kappa^{-1}(\tilde{y}) \subseteq X$ can be merged at their common zero-coordinate f to form a new point $z \in X$ with $z_{(-\infty,0]} = x_{(-\infty,0]}$ and $z_{[0,\infty)} = y_{[0,\infty)}$. By definition of κ one gets $\tilde{z} := \kappa(z)$ with $\tilde{z}_{(-\infty,0]} = \tilde{x}_{(-\infty,0]}$ and $\tilde{z}_{[0,\infty)} = \tilde{y}_{[0,\infty)}$, so f is in fact synchronizing for Y and every point in $Y \setminus \{\uparrow^{\infty}\}$ sees a synchronising symbol.
It remains to show that $\kappa|_{\mathrm{DT}(X_0)}$ is a topological conjugacy: As $X_0 = X \,\dot\cup\, \{\infty\}$ we have $\mathrm{DT}(X_0) = \mathrm{DT}(X)$. Every point $y \in \mathrm{DT}(Y)$ contains infinitely many edges from F in its left- and its right-infinite ray. The blocks $\uparrow^n$ $(n \in \mathbb{N})$ between those edges can be decoded uniquely to paths in G. There is an unique preimage $x \in \mathrm{DT}(X)$ with $\kappa(x) = y$. This proves bijectivity of $\kappa|_{\mathrm{DT}(X_0)}$.
Finally the inverse map $(\kappa|_{\mathrm{DT}(X_0)})^{-1}$ is continuous with respect to the induced topologies on $\mathrm{DT}(X_0) \subseteq X$ and $\mathrm{DT}(Y) \subseteq Y$: Let $y \in \mathrm{DT}(Y)$ and $W(x) \subseteq \mathrm{DT}(X)$ some neighbourhood of $x := \kappa^{-1}(y) \in X$. For $m,n \in \mathbb{N}$ large enough, $W(x)$ contains a cylinder ${}_{-n}[x_{-n} \dots x_0 \dots x_m] \cap \mathrm{DT}(X)$ with $x_{-n},\ x_m \in F$. Its image $V(y) := \kappa\big({}_{-n}[x_{-n} \dots x_0 \dots x_m] \cap \mathrm{DT}(X)\big)$ is compact-open, contains y and satisfies $\kappa^{-1}\big(V(y)\big) = {}_{-n}[x_{-n} \dots x_0 \dots x_m] \cap \mathrm{DT}(X) \subseteq W\big(\kappa^{-1}(y)\big) \subseteq \mathrm{DT}(X)$. □

4. On the subgroup structure of $\mathrm{Aut}(\sigma)$

Using marker constructions lots of abstract groups have been embedded into the automorphism groups of SFTs (see [**Hed**], [**BLR**], [**KR1**], [**Kit**]) to show their rich and diverse structure and to exhibit some algebraic restrictions. In this context we call an abstract group H a subgroup of $\mathrm{Aut}(\sigma)$, if $\mathrm{Aut}(\sigma)$ contains a subgroup isomorphic to H. Since it is possible to carry over the whole concept of marker automorphisms to countable state Markov shifts, all subgroups realized in the automorphism groups of SFTs also show up in the non-compact setting. Therefore according to [**BLR**] the automorphism group of every transitive (mixing), countable state Markov shift contains any direct sum of countably many finite groups, the direct sum of countably many copies of $\mathbb{Z}$, the free group on countably many generators and any free product of finitely many cyclic groups, as well as all of their subgroups. Moreover the fundamental group of any 2-manifold and any countable, locally finite, residually finite group is a subgroup in $\mathrm{Aut}(\sigma)$ ([**KR1**]).
What about other groups realizable only in the countable state case and what about remaining, relaxed and new restrictions and algebraic properties?

Recall from proposition 2.1 that – even in the non locally compact setting – the automorphism group of any transitive, countable state Markov shift is a subgroup of $\mathcal{S}_{\mathbb{N}}$. Using the work of N.G. de Bruijn as well as that of M. Kneser and S. Swierczkowski one can exclude certain abstract groups from being subgroups of $\mathrm{Aut}(\sigma)$: The group of all finite permutations on a set of cardinality $2^{\aleph_0}$ ([**Bru1**], theorem 5.1) and the group $H := {}^{F}/_{F''}$, where F is a non-abelian free group with more than $2^{\aleph_0}$ generators, F' its commutator subgroup and F'' the commutator group of F' ([**KS**], theorem 2), cannot be realized in the automorphism group of any transitive, countable state Markov shift (subshift with dense periodic points).

On the contrary there is at least a class of non locally compact, countable state Markov shifts with $\mathcal{S}_{\mathbb{N}}$ itself occuring as a subgroup in $\mathrm{Aut}(\sigma)$. The automorphism groups of this class are hence universal in the sense that they contain a copy of the automorphism group of any transitive, countable state subshift with periodic points dense (apply the argument in the proof of proposition 2.1). Prototype for this class is the full-shift $\mathcal{A}^{\mathbb{Z}}$ with $|\mathcal{A}| = \aleph_0$.

Proposition 4.1. *If a transitive, non locally compact, countable state Markov shift is presentable on a graph containing an infinite number of paths of fixed length connecting a common initial with a common terminal vertex, then $\mathcal{S}_{\mathbb{N}}$ is (isomorphic to) a subgroup of its automorphism group.*

Proof: Let $G = (V, E)$ be a graph presentation as assumed in the proposition, $k \in \mathbb{N}$ a shortest path length such that there are two vertices $u, v \in V$ ($u = v$ allowed) with infinitely many distinct paths p_i ($i \in \mathbb{N}_0$) of length k between them. W.l.o.g. assume all paths p_i pairwise edge-disjoint. This is possible, since due to the minimality of k any edge in E can only be part of a finite number of paths p_i. As G is strongly connected, there is a shortest path q connecting $v = \mathfrak{t}(p_i)$ with $u = \mathfrak{i}(p_i)$. Let $f \in E$ be the initial edge of p_0. For every permutation $\pi \in \mathcal{S}_{\mathbb{N}}$ define a map $\varphi_\pi : X_G \to X_G$ which scans a point and replaces every block $p_i\, q\, f$ with $p_{\pi(i)}\, q\, f$ ($i \in \mathbb{N}$). These are well-defined (f cannot occur in q or any p_i), bijective sliding-block-codes with memory and anticipation $\leq k + |q|$. Since

$\varphi_\pi \circ \varphi_\tau = \varphi_{\pi\circ\tau}$ and ${\varphi_\pi}^{-1} = \varphi_{\pi^{-1}}$, we have constructed a set of automorphisms $\{\varphi_\pi \mid \pi \in \mathcal{S}_\mathbb{N}\} \leq \mathrm{Aut}(\sigma)$ isomorphic to $\mathcal{S}_\mathbb{N}$. □

We collect the strong implications of proposition 4.1 in the following corollary. To say it in short: Most of the algebraic restrictions known for the automorphism groups of SFTs (see [**BLR**], section 3) vanish completely for the class of Markov shifts described above and a lot of subgroups which are forbidden for SFTs show up.

Recall that the automorphism group is residually finite, if for every element $\varphi \in \mathrm{Aut}(\sigma)$, $\varphi \neq \mathrm{Id}_X$ there is a finite group H and a homomorphism $\alpha: \mathrm{Aut}(\sigma) \to H$ with $\alpha(\varphi) \neq 1_H$. This excludes the existence of both infinite simple and nontrivial divisible subgroups.

An abstract group H is divisible, if for every element $h \in H$ and every $n \in \mathbb{N}$ there is an element $g \in H$ with $g^n = h$.

Corollary 4.2. *Let (X, σ) be a transitive, non locally compact, countable state Markov shift as in proposition 4.1.*

Its automorphism group contains infinite simple groups and is thus not residually finite. Every countable group can be realized in $\mathrm{Aut}(\sigma)$*. In particular the divisible groups $\mathbb{Q}$ and $\mathbb{Z}(p^\infty) = \mathbb{Z}[1/p]/\mathbb{Z}$ (p prime) can be embedded. The automorphism group does contain finitely generated groups with unsolvable word problem. Moreover every abelian group of cardinality $2^{\aleph_0}$ (especially $\mathbb{R}$) occurs in* $\mathrm{Aut}(\sigma)$*. Finally its set of subgroups is closed under taking free products of any $2^{\aleph_0}$ of its elements.*

Proof: The existence of the infinite simple subgroup $\mathcal{A}_{\mathbb{N},\mathfrak{f}} \leq \mathcal{S}_\mathbb{N}$ (alternating group on a countably infinite set) within $\mathrm{Aut}(\sigma)$ prohibits residual finiteness.

Every countable group H operates on itself by (left-)translation $\alpha_g: H \to H,\ h \mapsto g\,h \ \ \forall g \in H$. This yields a representation of H as a group of permutations on H. So $H \leq \mathcal{S}_H \cong \mathcal{S}_\mathbb{N} \leq \mathrm{Aut}(\sigma)$.

According to R. Lyndon and P. Schupp ([**LS**], theorem IV.7.2) there are finitely generated, countable groups with unsolvable word problem, e.g. $H := \langle a, b, c, d \mid a^{-i}\, b\, a^i = c^{-i}\, d\, c^i \text{ iff } i \in S \rangle$ where $S \subsetneq \mathbb{N}$ is a recursively enumerable, non recursive subset.

The last two statements follow from the work of N.G. de Bruijn ([**Bru2**], theorem 4.3 and [**Bru1**], theorem 4.2). □

Next we specify a larger class of transitive, (non) locally compact, countable state Markov shifts admitting at least an embedding of the restricted permutation group $\mathcal{S}_{\mathbb{N},\mathfrak{f}}$ into $\mathrm{Aut}(\sigma)$. For this we need a graph presentation containing a strongly connected, infinite, tree-like subgraph consisting of an infinite number of loops l_i ($i \in \mathbb{N}$) – the nodes of the tree – of uniform length and of paths $p_{i,j}$, $p_{j,i}$ – the links between the nodes – connecting the loops l_i and l_j. If the length of all paths $p_{i,j}$, $p_{j,i}$ in this tree-structure is bounded globally, we can construct a subgroup of $\mathrm{Aut}(\sigma)$ isomorphic to $\mathcal{S}_{\mathbb{N},\mathfrak{f}}$. Obviously this class contains the previously considered family of non locally compact Markov shifts (proposition 4.1) as well as a subclass of locally compact, countable state Markov shifts. Prototypes for this class are (topological) random walks on $\mathbb{N}$ or on $\mathbb{Z}$ with steps $0, \pm 1$.

Proposition 4.3. *If any graph presentation of a transitive, countable state Markov shift contains an infinite set of loops $L = \{l_i \mid i \in \mathbb{N}\}$ of equal length, such that for every loop l_i there is, within a bounded distance, another loop l_j ($i < j \in \mathbb{N}$), i.e. there is a path $p_{i,j}$ connecting a vertex of l_i with one of l_j, a path $p_{j,i}$ connecting l_j back to l_i and both paths have length bounded by a global constant, then $\mathcal{S}_{\mathbb{N},\mathfrak{f}}$ can be embedded into the automorphism group.*

Proof: We distinguish between two cases: Either the graph presentation $G = (V, E)$, as assumed in the proposition, contains a loop $l \in L$ and an infinite subset $L' \subseteq L$ of loops having distance to l bounded by some constant $M \in \mathbb{N}$. Then G cannot be locally finite. G already fulfills the assumptions of proposition 4.1, because using the elements in L' there are infinitely many distinct paths of length $\leq 3(|l| - 1) + 2M$ from one vertex in l back to this vertex. Thus not only $\mathcal{S}_{\mathbb{N},\mathfrak{f}}$ but even $\mathcal{S}_{\mathbb{N}}$ can be embedded into $\mathrm{Aut}(\sigma)$.
In the remaining case the tree-like subgraph consisting of the loops $l_i \in L$ and the paths $p_{i,j},\ p_{j,i}$ is locally finite. After renumbering we can find an infinite chain of loops l_k ($k \in \mathbb{N}$) being connected via paths $p_{k,k+1}$ and $p_{k+1,k}$ of length bounded by $M \in \mathbb{N}$. W.l.o.g. choose $p_{k,k+1},\ p_{k+1,k}$ minimal, such that $\mathfrak{i}(p_{1,2}) = \mathfrak{t}(p_{2,1}) = \mathfrak{i}(l_1)$ and $\mathfrak{t}(p_{k,k+1}) = \mathfrak{i}(p_{k+1,k}) = \mathfrak{i}(p_{k+1,k+2}) = \mathfrak{t}(p_{k+2,k+1}) = \mathfrak{i}(l_{k+1})$ for all $k \in \mathbb{N}$. For the rest of the proof it suffices to look at this linear chain.
Let $N := |l_k|$ be the common length of all loops l_k. Define a countably infinite set of closed paths

$$b_k := p_{k,k+1}\, p_{k+1,k}\, l_k\, (p_{k,k+1}\, p_{k+1,k})^{\frac{(2M)!}{|p_{k,k+1} p_{k+1,k}|} - 1} \qquad (k \in \mathbb{N})$$

of uniform length $(2M)! + N$ which, due to the minimality of $p_{k,k+1},\ p_{k+1,k}$, allow no nontrivial overlaps. Furthermore cyclically shifting the blocks b_k by $|p_{k,k+1}|$ symbols to the left yields

$$\tilde{b}_k := p_{k+1,k}\, l_k\, (p_{k,k+1}\, p_{k+1,k})^{\frac{(2M)!}{|p_{k,k+1} p_{k+1,k}|} - 1}\, p_{k,k+1} \qquad (k \in \mathbb{N})$$

Obviously $\mathfrak{i}(\tilde{b}_k) = \mathfrak{t}(\tilde{b}_k) = \mathfrak{i}(b_{k+1}) = \mathfrak{t}(b_{k+1})$ and $|\tilde{b}_k| = |b_k| = (2M)! + N$ for all $k \in \mathbb{N}$.
For every $k \in \mathbb{N}$ define a $\big((2M)! + N - 1, (2M)! + N - 1\big)$-sliding-block-code $\phi_{(k,k+1)} : X \to X$, which scans a point and replaces every block $\tilde{b}_k$ by b_{k+1} as well as every block b_{k+1} by $\tilde{b}_k$. Since $\tilde{b}_k$ and b_{k+1} cannot overlap, $\phi_{(k,k+1)}$ is well-defined. By definition these maps are continuous, shiftcommuting involutions, hence automorphisms. Moreover $\phi_{(k,k+1)}\big((\tilde{b}_k)^\infty\big) = (b_{k+1})^\infty$ and $\phi_{(k,k+1)}\big((b_{k+1})^\infty\big) = (\tilde{b}_k)^\infty$ imply $\phi_{(k,k+1)}\big(\mathrm{Orb}\big((b_k)^\infty\big)\big) = \mathrm{Orb}\big((b_{k+1})^\infty\big)$, $\phi_{(k,k+1)}\big(\mathrm{Orb}\big((b_{k+1})^\infty\big)\big) = \mathrm{Orb}\big((b_k)^\infty\big)$ and $\phi_{(k,k+1)}\big(\mathrm{Orb}\big((b_i)^\infty\big)\big) = \mathrm{Orb}\big((b_i)^\infty\big)$ for all $i \neq k, k+1$. The family of automorphisms $\big(\phi_{(k,k+1)}\big)_{k\in\mathbb{N}}$ acts on $\mathcal{O} := \big\{\mathrm{Orb}\big((b_k)^\infty\big) \mid k \in \mathbb{N}\big\}$ like the set of transpositions $\big((k, k+1)\big)_{k\in\mathbb{N}}$ does on $\mathbb{N}$. One easily checks that different presentations of a finite permutation on $\mathcal{O}$ as finite products of the $\phi_{(k,k+1)}$ yield the same automorphism. As any permutation in $\mathcal{S}_{\mathbb{N},\mathfrak{f}} \cong \langle (k, k+1) \mid k \in \mathbb{N}\rangle$ is presentable as a finite product of transpositions, the set $\big\{\phi_{(k,k+1)} \mid k \in \mathbb{N}\big\}$ generates a subgroup of $\mathrm{Aut}(\sigma)$ isomorphic to $\mathcal{S}_{\mathbb{N},\mathfrak{f}}$. □

Corollary 4.4. *The automorphism groups of topological Markov shifts satisfying the assumptions of proposition 4.3 contain infinite simple subgroups and are thus not residually finite.*

Proof: The alternating group $\mathcal{A}_{\mathbb{N},\mathfrak{f}}$ on a countably infinite set is an infinite simple subgroup of $\mathcal{S}_{\mathbb{N},\mathfrak{f}}$. Hence $\mathrm{Aut}(\sigma) \geq \mathcal{A}_{\mathbb{N},\mathfrak{f}}$ is not residually finite. □

Finally we show that property **(FMDP)** implies all of the restrictions on the subgroup structure of $\mathrm{Aut}(\sigma)$ known for SFTs:

Theorem 4.5. *For every transitive, locally compact, countable state Markov shift property* **(FMDP)** *forces the existence of a formal zetafunction, i.e. for any given period there are only finitely many periodic points.*

Proof: Let $G = (V, E)$ be a strongly connected, locally finite graph presenting the Markov shift and let $F \subsetneq E$ be a finite set of edges such that every doublepath in G (having **(FMDP)**) contains an element from F. Suppose X_G has no formal zetafunction.
There is a smallest period length $k \in \mathbb{N}$ with $|\mathrm{Per}_k(X_G)| = \aleph_0$ and G has infinitely many simple loops of length k. (A path/loop is called simple, if it has no proper closed subpath.) As G is locally finite, one can choose an infinite set $L := \{l_i \mid i \in \mathbb{N}_0\}$ of such simple loops that are pairwise vertex-disjoint and in addition edge-disjoint from the set F. For every $i \in \mathbb{N}$ choose a shortest path p_i from $\mathfrak{i}(l_0)$ to $\mathfrak{i}(l_i)$ and a shortest path q_i from $\mathfrak{i}(l_i)$ back to $\mathfrak{i}(l_0)$. Since $l_0\, p_i$ and $p_i\, l_i$ form a doublepath, p_i has to contain an edge from F. The same is true for q_i. Using a pigeon hole argument, one gets a pair of subsets M_1, $M_2 \subseteq F$ such that there exists an infinite subset:

$$L' := \{l_i \in L \mid i \in \mathbb{N} \,\wedge\, (f \in M_1 \Leftrightarrow f \in p_i) \,\wedge\, (f \in M_2 \Leftrightarrow f \in q_i)\} \subseteq L$$

For notational simplicity renumber the elements in $L' = \{l_i \mid i \in \mathbb{N}\}$ (as well as their paths p_i, q_i) consecutively.
By construction the elements in M_1 occur exactly once and in an uniform order in all paths p_i. Analogously for M_2 and q_i. Look at the shortened paths $\tilde{p}_i$ being the suffix of p_i, connecting the terminal vertex of the last edge from M_1 with $\mathfrak{i}(l_i)$ and $\tilde{q}_i$ being the prefix of q_i connecting $\mathfrak{i}(l_i)$ to the initial vertex of the first edge from M_2. Obviously no $\tilde{p}_i$, $\tilde{q}_i$ does contain an edge from F, but all of them start (end) at a common vertex. Another pigeon hole argument gives two distinct indices $i \neq j \in \mathbb{N}$ such that $|\tilde{p}_i| + |\tilde{q}_i| = |\tilde{p}_j| + |\tilde{q}_j| + m \cdot k$ with $m \in \mathbb{N}_0$. The doublepath $[\tilde{p}_i\, \tilde{q}_i; \tilde{p}_j\, {l_j}^m\, \tilde{q}_j]$ contradicts the assumption on F. Therefore X_G has a formal zetafunction. □

Theorem 4.5 allows us to get most of the restrictive results on the algebraic structure of the automorphism groups of SFTs from section 3 in [**BLR**] by simply copying the proofs using only the existence of a zetafunction.

Corollary 4.6. *Let* (X, σ) *be a transitive, locally compact, countable state Markov shift with* $\mathrm{Aut}(\sigma)$ *countable. Then the automorphism group is residually finite. Thus* $\mathrm{Aut}(\sigma)$ *neither contains any nontrivial divisible nor any infinite simple subgroup. This excludes some abstract countable (abelian) groups, like* $\mathcal{A}_{\mathbb{N},\mathfrak{f}}$, $\mathrm{PSL}_n(\mathbb{Q})$ *(the projective unimodular groups over the rationals for* $2 \leq n \in \mathbb{N}$*),* $\mathbb{Q}$, $\mathbb{Z}(p^\infty)$ *(p prime). A subgroup of* $\mathbb{Q}/\mathbb{Z}$ *is realized in* $\mathrm{Aut}(\sigma)$ *iff it is residually finite.*

Open Problem: After all these similarities between the automorphism groups of SFTs and countable state Markov shifts with property **(FMDP)** – both are countably infinite, residually finite groups with a seemingly equal subgroup structure, being discrete with respect to the compact-open topology and having the same center (see section 5) – we may ask the question whether all countable automorphism groups that show up for transitive, locally compact, countable state Markov shifts are already realized for transitive SFTs. Unfortunately up to now we do not know of any property that distinguishes between the automorphism groups of these two subshift-classes.

The results obtained so far already give a coarse classification of all transitive, countable state Markov shifts (X, σ) via their automorphism groups into 5 mutually disjoint, conjugacy-invariant classes:

	(X, σ) non locally compact	(X, σ) locally compact
$\mathrm{Aut}(\sigma)$ uncountable, non residually finite	very weak restrictions on algebraic properties and subgroups; e.g. subshifts satisfying proposition 4.1	weak restrictions, due to the absense of a zetafunction and of **(FMDP)**; e.g. locally-compact subshifts satisfying proposition 4.3
$\mathrm{Aut}(\sigma)$ uncountable, residually finite	no nontrivial divisible, no infinite simple subgroups; e.g. non locally compact, countable state Markov shifts with formal zetafunction	no nontrivial divisible, no infinite simple subgroups; examples can be constructed from graph presentations of transitive, locally compact, countable state Markov shifts with formal zetafunctions by doubling (n-folding) all edges
$\mathrm{Aut}(\sigma)$ countable, thus residually finite	not existent !	strong restrictions like in the SFT case; this class contains exactly the transitive, locally compact, countable state Markov shifts with **(FMDP)**

5. Ryan's theorem for countable state Markov shifts

As we have seen in the previous section, it is difficult to describe the automorphism groups of topological Markov shifts as abstract groups. Thus we look for further group-theoretic properties describing $\mathrm{Aut}(\sigma)$ and limiting the set of possible groups. One such property examined for SFTs is the center $\mathfrak{Z} = \mathfrak{Z}(\mathrm{Aut}(\sigma))$. J. Ryan ([**Rya1**] and [**Rya2**]) proved that for all transitive SFTs the center consists exactly of the powers of the shift map and is therefore (for all nontrivial, transitive SFTs) isomorphic to $\mathbb{Z}$.

Since by definition σ has to commute with every element in $\mathrm{Aut}(\sigma)$, we get $\{\sigma^i \mid i \in \mathbb{Z}\} \leq \mathfrak{Z}$ not just for Markov shifts but for any subshift (X, σ). Therefore the automorphism group of any nontrivial, transitive subshift has to have a center containing $\mathbb{Z}$ as a subgroup. Moreover the center is a normal subgroup in $\mathrm{Aut}(\sigma)$. This excludes certain abstract groups from being realized as automorphism groups of subshifts. For example:

PROPOSITION 5.1. *The automorphism group of any transitive, countable state Markov shift (nontrivial subshift) is not isomorphic to either $\mathcal{S}_{\mathbb{N}}$ or $\mathcal{S}_{\mathbb{N},\mathfrak{f}}$.*

PROOF: Suppose $\mathrm{Aut}(\sigma) \cong \mathcal{S}_{\mathbb{N}}$. The theorem of J. Schreier and S. Ulam [**SU**] gives the Jordan-Hölder decomposition $\mathcal{S}_{\mathbb{N}} \rhd \mathcal{S}_{\mathbb{N},\mathfrak{f}} \rhd \mathcal{A}_{\mathbb{N},\mathfrak{f}} \rhd \{1\}$ (factor groups being simple). Therefore $\mathfrak{Z} \trianglelefteq \mathrm{Aut}(\sigma)$ has to be isomorphic to one of these normal subgroups. Obviously this contradicts $\mathfrak{Z} \geq \{\sigma^i \mid i \in \mathbb{Z}\}$ being abelian.
The same argument shows $\mathrm{Aut}(\sigma) \not\cong \mathcal{S}_{\mathbb{N},\mathfrak{f}}$. □

After some preliminaries we can reprove Ryan's theorem for countable state Markov shifts:

LEMMA 5.2. *Every automorphism of some transitive Markov shift acting trivially on the set of (periodic) σ-orbits is a power of the shift map.*

PROOF: It suffices to show that any automorphism $\varphi \in \mathrm{Aut}(\sigma)$ of the transitive Markov shift (X, σ) inducing the identity on the set of periodic σ-orbits just shifts all periodic points of large-enough period by a common amount. Since every point in X can be approximated by a sequence of periodic points of large period, this already fixes the action of φ on all of X and proves φ being some power of σ.
Choose a periodic point $x \in \mathcal{O}_1$ from some minimal σ-orbit $\mathcal{O}_1 \subseteq X$ and let $N_1 \in \mathbb{N}$ be the orbit length of $\mathcal{O}_1$. Then the block $l_1 := x_{[0,N_1)} \in \mathcal{B}_{N_1}(X)$ defines x and cannot overlap itself nontrivially. Now $\varphi(x) = \sigma^{s_1}(x)$ for $-\frac{1}{2}N_1 < s_1 \leq \frac{1}{2}N_1$ uniquely determined. As φ is continuous, mapping all finite σ-orbits onto itself, there is a coding length $n_1 \in \mathbb{N}$ such that $\big(\varphi(y)\big)_{[0,N_1)} = \big(\sigma^{s_1}(x)\big)_{[0,N_1)}$ for all $y \in {}_{-n_1 N_1}[{l_1}^{2n_1}]$. Let $\mathcal{O}_2$, $\mathcal{O}_3$ be two distinct σ-orbits of lengths N_2, $N_3 \in \mathbb{N}$ larger than $2(n_1+1)N_1$ and let $l_2 \in \mathcal{B}_{N_2}(X)$, $l_3 \in \mathcal{B}_{N_3}(X)$ be defining blocks for $\mathcal{O}_2$, $\mathcal{O}_3$. Once again one has $\varphi({l_2}^\infty) = \sigma^{s_2}({l_2}^\infty)$ and $\varphi({l_3}^\infty) = \sigma^{s_3}({l_3}^\infty)$ for unique $-\frac{1}{2}N_2 < s_2 \leq \frac{1}{2}N_2$ and $-\frac{1}{2}N_3 < s_3 \leq \frac{1}{2}N_3$. Moreover there are numbers n_2, $n_3 \in \mathbb{N}$ for which $\big(\varphi(y)\big)_{[0,N_i)} = \big(\sigma^{s_i}({l_i}^\infty)\big)_{[0,N_i)}$ whenever $y \in {}_{-n_i N_i}[{l_i}^{2n_i}]$ $(i := 2, 3)$.
Using the irreducibility of X one can find blocks $p_{12}, p_{23}, p_{31} \in \mathcal{B}(X)$ of minimal length such that $l_1\, p_{12}\, l_2\, p_{23}\, l_3\, p_{31}\, l_1 \in \mathcal{B}(X)$ is admissible for X. For $m \in \mathbb{N}$ with $m\, N_1 > \max\{|l_3\, p_{31}\, l_1\, p_{12}\, l_2|\, , |l_2\, p_{23}\, l_3|\}$ the periodic point $y := ({l_1}^{2n_1+m}\, p_{12}\, {l_2}^{2n_2}\, p_{23}\, {l_3}^{2n_3}\, p_{31})^\infty \in X$ has least period $M := (2n_1+m)N_1 + 2(n_2N_2 + n_3N_3) + |p_{12}\, p_{23}\, p_{31}|$. In particular a block ${l_1}^m$ can only occur inside ${l_1}^{2n_1+m}$. As before $\varphi(y) = \sigma^s(y)$ for $-\frac{1}{2}M < s \leq \frac{1}{2}M$ unique.
Now $\big(\sigma^{s+n_1N_1}(y)\big)_{[0,(m+1)N_1)} = \big(\sigma^{s_1}(x)\big)_{[0,(m+1)N_1)}$ guarantees $-(n_1+1)N_1 < s \leq (n_1+1)N_1$. Since $N_2 > 2(n_1+1)N_1$, this implies $-\frac{1}{2}N_2 < s \leq \frac{1}{2}N_2$. But then $s = s_2$, because using the coding length of the block l_2 one gets $\big(\sigma^{s+(2n_1+m)N_1+|p_{12}|+n_2N_2}(y)\big)_{[0,N_2)} = \big(\sigma^{s_2}({l_2}^\infty)\big)_{[0,N_2)}$. As $N_3 > 2(n_1+1)N_1$, the same argument shows $s = s_3$. Therefore all periodic σ-orbits of length greater than $2(n_1+1)N_1$ are shifted under φ by the same amount. □

REMARK: The proof of lemma 5.2 merely relies on the Markov property and the transitivity, but not on the cardinality of the alphabet. Hence it can be used for SFTs, countable state (and even larger) Markov shifts.

Using the periodic-orbit representation of the automorphism group, originally introduced for SFTs by M. Boyle and W. Krieger (see [**BK1**] or [**BLR**]), we can translate lemma 5.2 into the language of group theory. To achieve this we define the periodic-orbit representation ρ for automorphism groups of (countable state) Markov shifts exactly as for SFTs:
Let $\mathrm{Orb}_n(X) := {}^{\mathrm{Per}^0_n(X)}/_{\langle\sigma\rangle}$ the set of σ-orbits of length $n \in \mathbb{N}$. Then

$$\rho : \mathrm{Aut}(\sigma) \to \prod_{n=1}^{\infty} \mathrm{Aut}\big(\mathrm{Orb}_n(X), \sigma\big), \quad \varphi \mapsto \rho(\varphi) := \big(\rho_n(\varphi)\big)_{n\in\mathbb{N}}$$

where $\rho_n(\varphi) \in \mathrm{Aut}\big(\mathrm{Orb}_n(X), \sigma\big)$ is the permutation on the set of σ-orbits $\mathrm{Orb}_n(X)$ induced by $\varphi|_{\mathrm{Per}^0_n(X)}$. $\rho_n(\varphi)$ is well-defined, since $\varphi|_{\mathrm{Per}^0_n(X)} \in \mathrm{Aut}\left(\mathrm{Per}^0_n(X), \sigma\right)$ commutes with $\sigma|_{\mathrm{Per}^0_n(X)}$ and $\rho_n(\sigma) = \mathrm{Id}_{\mathrm{Orb}_n(X)}$.

Corollary 5.3. *For every transitive Markov shift (X, σ) the periodic-orbit representation of $\mathrm{Aut}(\sigma)$ is faithful on the group ${}^{\mathrm{Aut}(\sigma)}/_{\langle\sigma\rangle}$.*

Proof: Let $\varphi \in \mathrm{Aut}(\sigma)$ with $\rho(\varphi) = \mathrm{Id}$. Then $\rho_n(\varphi) = \mathrm{Id}_{\mathrm{Orb}_n(X)}$ for all $n \geq 1$. Lemma 5.2 implies $\varphi \in \langle\sigma\rangle$, hence $\varphi \in [\mathrm{Id}] \in {}^{\mathrm{Aut}(\sigma)}/_{\langle\sigma\rangle}$. □

Theorem 5.4. *The center of the automorphism group of any transitive (countable state) Markov shift consists exactly of the powers of the shift map.*

Proof: Suppose the automorphism $\varphi \in \mathrm{Aut}(\sigma)$ is no power of the shift map. Following from lemma 5.2 there are two distinct (periodic) σ-orbits $\mathcal{O}_1$, $\mathcal{O}_2$ of some length $N \in \mathbb{N}$ such that $\varphi(\mathcal{O}_1) = \mathcal{O}_2$. Let $x^{(1)} \in \mathcal{O}_1$, $x^{(2)} := \varphi(x^{(1)}) \in \mathcal{O}_2$ and $x^{(3)} := \varphi(x^{(2)}) \in \mathrm{Per}^0_N(X)$. By continuousity, the blocks $l_i := (x^{(i)})_{[0,N)} \in \mathcal{B}_N(X)$ $(i := 1, 2, 3)$ satisfy $\varphi({}_{-m_1N}[{l_1}^{2m_1}]) \subseteq {}_0[l_2]$ und $\varphi({}_{-m_2N}[{l_2}^{2m_2}]) \subseteq {}_0[l_3]$ for m_1, $m_2 \in \mathbb{N}$ large enough.
Define a periodic point $y^{(1)} := ({l_1}^{m+m_1}\, p_{12}\, {l_2}^{m_2}\, p_{21}\, {l_1}^{m_3}\, \tilde{p}_{12}\, {l_2}^{m_4}\, \tilde{p}_{21}\, {l_1}^{m_5})^\infty \in X$ of large period, where p_{12}, p_{21}, $\tilde{p}_{12}$, $\tilde{p}_{21} \in \mathcal{B}(X)$ are non empty blocks not containing a complete block l_1, l_2 or l_3 (those exist, since X is irreducible). Furthermore $m_i \in \mathbb{N}$ $(1 \leq i \leq 5)$ are chosen large enough to get an image of the form $y^{(2)} := \varphi(y^{(1)}) = ({l_2}^{m+n_1}\, q_{23}\, {l_3}^{n_2}\, q_{32}\, {l_2}^{n_3}\, \tilde{q}_{23}\, {l_3}^{n_4}\, \tilde{q}_{32}\, {l_2}^{n_5})^\infty$ with $n_i \geq 1$ $(1 \leq i \leq 5)$. As suggested by this representation no prefix of q_{23}, $\tilde{q}_{23}$ and no suffix of q_{32}, $\tilde{q}_{32}$ shall be a complete block l_2; no prefix of q_{32}, $\tilde{q}_{32}$ and no suffix of q_{23}, $\tilde{q}_{23}$ is a complete block l_3. Finally m_2, m_4 can be tuned to get $n_2 \neq n_4$ and $m \in \mathbb{N}$ should satisfy $m\,N > (m_1 + m_2 + m_3 + m_4 + m_5)\,N + |p_{12}\, p_{21}\, \tilde{p}_{12}\, \tilde{p}_{21}|$. This guarantees $y^{(1)}$, $y^{(2)} \in \mathrm{Per}^0_M(X)$ with $M := |p_{12}\, p_{21}\, \tilde{p}_{12}\, \tilde{p}_{21}| + N\,(m+m_1+m_2+m_3+m_4+m_5)$; the block $(y^{(2)})_{[0,M)}$ has only trivial overlaps to itself.
Thus define an involutoric sliding-block-code $\psi :\ X \to X$ interchanging the blocks ${l_2}^{m+n_1}\, q_{23}\, {l_3}^{n_2}\, q_{32}\, {l_2}^{n_3}\, \tilde{q}_{23}\, {l_3}^{n_4}\, \tilde{q}_{32}\, {l_2}^{n_5}$ and ${l_2}^{m+n_1}\, q_{23}\, {l_3}^{n_4}\, q_{32}\, {l_2}^{n_3}\, \tilde{q}_{23}\, {l_3}^{n_2}\, \tilde{q}_{32}\, {l_2}^{n_5}$; ${l_2}^{m+n_1}\, q_{23}$ and $\tilde{q}_{32}\, {l_2}^{n_5}$ acting as markers. So $\psi(y^{(1)}) = y^{(1)}$, but $\psi(y^{(2)}) \neq y^{(2)}$. Obviously $\psi \in \mathrm{Aut}(\sigma)$ does not commute with φ, since $(\varphi \circ \psi)(y^{(1)}) = \varphi(y^{(1)}) = y^{(2)} \neq \psi(y^{(2)}) = (\psi \circ \varphi)(y^{(1)})$. So $\varphi \notin \mathfrak{Z}$ and $\mathfrak{Z} = \big\{\sigma^i \bigm| i \in \mathbb{Z}\big\}$. □

Concerning the center of $\mathrm{Aut}(\sigma)$ we get the same restrictions for (countable state) Markov shifts as for SFTs. Theorem 5.4 additionally can be used to exclude further abstract groups from being realized as automorphism groups of transitive Markov shifts. In particular the comparison between Markov shifts and coded systems yields considerably differences. Recalling some of the results by D. Fiebig and U.-R. Fiebig presented in [**FF2**], there are coded systems with automorphism groups isomorphic to any infinite, finitely generated abelian group (e.g. $\mathrm{Aut}(\sigma) \cong \langle\sigma\rangle \oplus \mathbb{Z}$) as well as isomorphic to $\langle\sigma\rangle \oplus \mathbb{Z}[1/2]$ or $\langle\sigma\rangle \oplus G$ with $G \leq \mathbb{Q}/\mathbb{Z}$ residually finite. Of course non of these can occur for transitive Markov shifts. Another result from [**FF2**] proves the existence of a coded system (X, σ_X) with automorphism group $\mathrm{Aut}(\sigma_X) \cong \langle\sigma_X\rangle \oplus \mathrm{Aut}(\sigma_Y)$, where (Y, σ_Y) is any nontrivial subshift with periodic points dense – a situation completely impossible for transitive Markov shifts, unless $\mathfrak{Z}(\mathrm{Aut}(\sigma_Y))$ is trivial, which is only true for a system of fixed points.

It seems that the automorphism groups of coded systems and countable state Markov shifts differ a lot, whereas those of SFTs and countable state Markov shifts are quite similar (leaving alone the possibility of new classes of subgroups, which is due to the enlarged alphabet). The remaining amount of compactness influences the size – cardinality and possible subgroups – of $\mathrm{Aut}(\sigma)$, but its fundamental structure is governed by the Markov property (plenty of marker automorphisms leading to the same result on the center of $\mathrm{Aut}(\sigma)$; similarities between automorphism groups of SFTs and countable state Markov shifts with **(FMDP)**).

Acknowledgement. This paper constitutes a part of the authors dissertation, which was prepared at the University of Heidelberg under the auspices of Prof. W. Krieger. The author would like to thank him for his constant support and some useful suggestions.
A great deal of gratitude also goes to PD U.-R. Fiebig who stayed at Heidelberg until March 2002, for initiating this work and for many fruitful discussions.

References

[**BK1**] Boyle, M. and Krieger, W., *Periodic points and automorphisms of the shift*, Trans. Am. Math. Soc. **302**, 125-149 (1987).

[**BK2**] Boyle, M. and Krieger, W., *Automorphisms and subsystems of the shift*, J. Reine Angew. Math. **437**, 13-28 (1993).

[**BLR**] Boyle, M., Lind, D. and Rudolph, D., *The automorphism group of a shift of finite type*, Trans. Am. Math. Soc. **306**, No.1, 71-114 (1988).

[**Bru1**] Bruijn, N.G. de, *Embedding theorems for infinite groups*, Nederl. Akad. Wet., Proc., Ser. **A 60**, 560-569 (1957).

[**Bru2**] Bruijn, N.G. de, *Addendum to 'Embedding theorems for infinite groups'*, Nederl. Akad. Wet., Proc., Ser. **A 67**, 594-595 (1964).

[**FieD**] Fiebig, D., *Factor maps, entropy and fiber cardinality for Markov shifts*, Rocky Mt. J. Math. **31**, No.3, 955-986 (2001).

[**FF1**] Fiebig, D. and Fiebig, U.-R., *Topological boundaries for countable state Markov shifts*, Proc. Lond. Math. Soc., Ser. **III. 70**, No.3, 625-643 (1995).

[**FF2**] Fiebig, D. and Fiebig, U.-R., *The automorphism group of a coded system*, Trans. Am. Math. Soc. **348**, No.8, 3173-3191 (1996).

[**FieU1**] Fiebig, U.-R., *Gyration numbers for involutions of subshifts of finite type I*, Forum Math. **4**, No.1, 77-108 (1992).

[**FieU2**] Fiebig, U.-R., *Gyration numbers for involutions of subshifts of finite type II*, Forum Math. **4**, No.2, 187-215 (1992).

[**Hed**] Hedlund, G., *Endomorphisms and automorphisms of the shift dynamical system*, Math. Systems Theory **3**, 320-375 (1969).

[KR1] Kim, K.H. and Roush, F.W., *On the automorphism groups of subshifts*, PU.M.A., Pure Math. Appl., Ser. **B 1**, No.4, 203-230 (1990).

[KR2] Kim, K.H. and Roush, F.W., *On the structure of inert automorphisms of subshifts*, PU.M.A., Pure Math. Appl., Ser. **B 2**, No.1, 3-22 (1991).

[KRW1] Kim, K.H., Roush, F.W. and Wagoner, J., *Automorphisms of the dimension group and gyration numbers*, J. Am. Math. Soc. **5**, No.1, 191-212 (1992).

[KRW2] Kim, K.H., Roush, F.W. and Wagoner, J., *The shift equivalence problem*, Math. Intell. **21**, No.4, 18-29 (1999).

[Kit] Kitchens, B., *Symbolic dynamics. One-sided, two-sided and countable state Markov shifts*, Universitext, Springer Verlag (1998).

[KS] Kneser, M. and Swierczkowski, S., *Embeddings in groups of countable permutations*, Colloq. Math. **7**, 177-179 (1960).

[LM] Lind, D. and Marcus, B., *An introduction to symbolic dynamics and coding*, Cambridge University Press (1995).

[LS] Lyndon, R. and Schupp, P., *Combinatorial group theory*, Classics in Mathematics, Springer Verlag (2001).

[Rya1] Ryan, J., *The shift and commutativity*, Math. Systems Theory **6**, 82-85 (1972).

[Rya2] Ryan, J., *The shift and commutativity II*, Math. Systems Theory **8**, 249-250 (1975).

[Sch] Schraudner, M., *Über die Automorphismengruppen topologischer Markovshifts mit abzählbar unendlicher Zustandsmenge*, Ph.D. thesis, Universität Heidelberg (2004). Online-publication: http://www.ub.uni-heidelberg.de/archiv/5097

[SU] Schreier, J. and Ulam, S., *Über die Permutationsgruppe der natürlichen Zahlenfolge*, Stud. Math. **4**, 134-141 (1933).