Entropy and mixing for $\mathbb{Z}^d$ SFTs

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- Such measures are useful for studying topological entropy, since they allow the additional strength of measure theory to be brought to bear

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- Same conditional probabilities if $\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & & & 0 \\ 0 & & & 1 \\ 1 & 0 & 0 & 0 \end{array}$ changed to $\begin{array}{cccccc} & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & \\ 1 & 0 & & & 0 & 0 \\ & 0 & & & 1 & \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & & & 1 & \end{array}$

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  - We'll return to this

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- This implies the claim from Lecture 1 that strongly irreducible SFTs are entropy minimal (have no proper subshifts with equal entropy)

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    - It's possible to transition from a positive boundary condition to a negative letter at 0 (strong irreducibility), but very unlikely (multiple uniform Gibbs measures)

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- Differs from usual notion of measure-theoretic mixing due to (necessary) idea of "surrounding"

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- More about this next time