Entropy and mixing for $\mathbb{Z}^d$ SFTs

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Iceberg model

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  - Then $\mu^+ \neq \mu^-$; they give different values to set $\bigcup_{i=1}^{M} [x(0) = i]$

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- $[x(0) > 0] = \bigcup_S E_S$, and the union is disjoint

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- Therefore, $\mu(E_S \mid x(\partial\{-n,\ldots,n\}^d) = -M) < M^{-|S|}$

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- Can be made smaller than $\frac{1}{2}$ for large M

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- So iceberg model has multiple MMEs for large M
- Even though it is as topologically mixing as possible (strong irreducibility), measure-theoretically it is badly nonmixing
- How does one prove uniqueness/spatial mixing for an SFT?
- Will outline for hard square shift $\mathcal{H}$ by nice argument due to van den Berg and Steif

Hard square shift

- **Lemma:** (van den Berg-Steif) If μ and μ' are uniform Gibbs measures for an SFT X, and if $(\mu \times \mu')$(there exists an infinite path P on which x, y disagree) $= 0$, then $\mu = \mu'$.

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- $\phi([w] \times [w]^c) = [w]^c \times [w]$, so $(\mu \times \mu')([w] \times [w]^c) = (\mu \times \mu')([w]^c \times [w])$

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- Since w was arbitrary, $\mu = \mu'$

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  - $(\mu \times \mu')(x(0) \neq y(0) \mid E)$ can be written as a weighted average of $(\mu \times \mu')(x(0) \neq y(0) \mid x(\{\pm e_i\}) = \delta, y(\{\pm e_i\}) = \delta')$

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- Relates problem of uniqueness of MME to percolation theory

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- As explained earlier, this implies computability of $h(\mathcal{H})$ in polynomial time
- More refined techniques imply more general results
- **Theorem:** (P.) There exists ϵ such that for any nearest neighbor $\mathbb{Z}^2$ SFT X with $h(X) > \log |A| - \epsilon$, X has a unique MME, and $h(X)$ is computable in polynomial time