An algebraic geometric approach to multidimensional words

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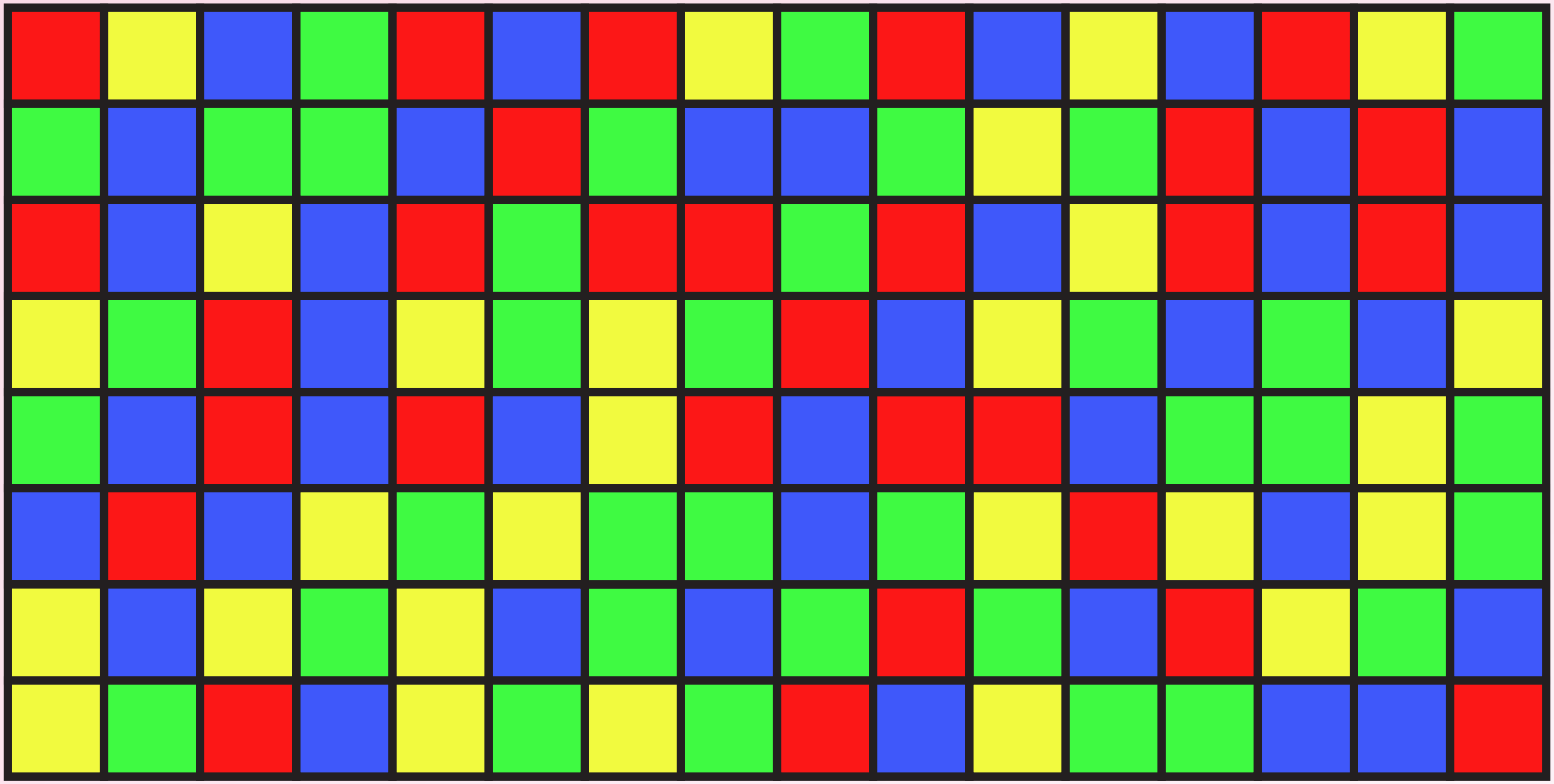
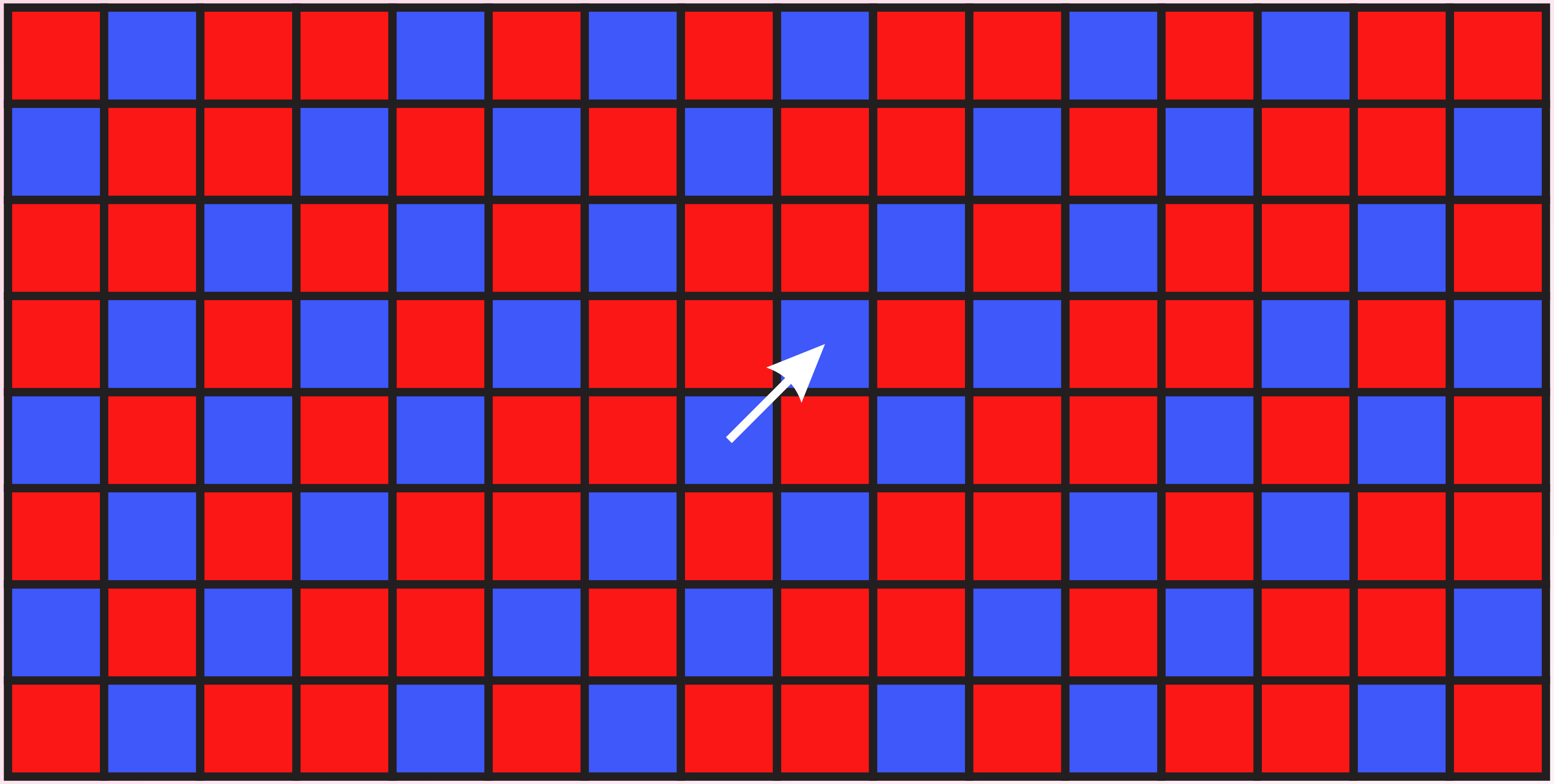
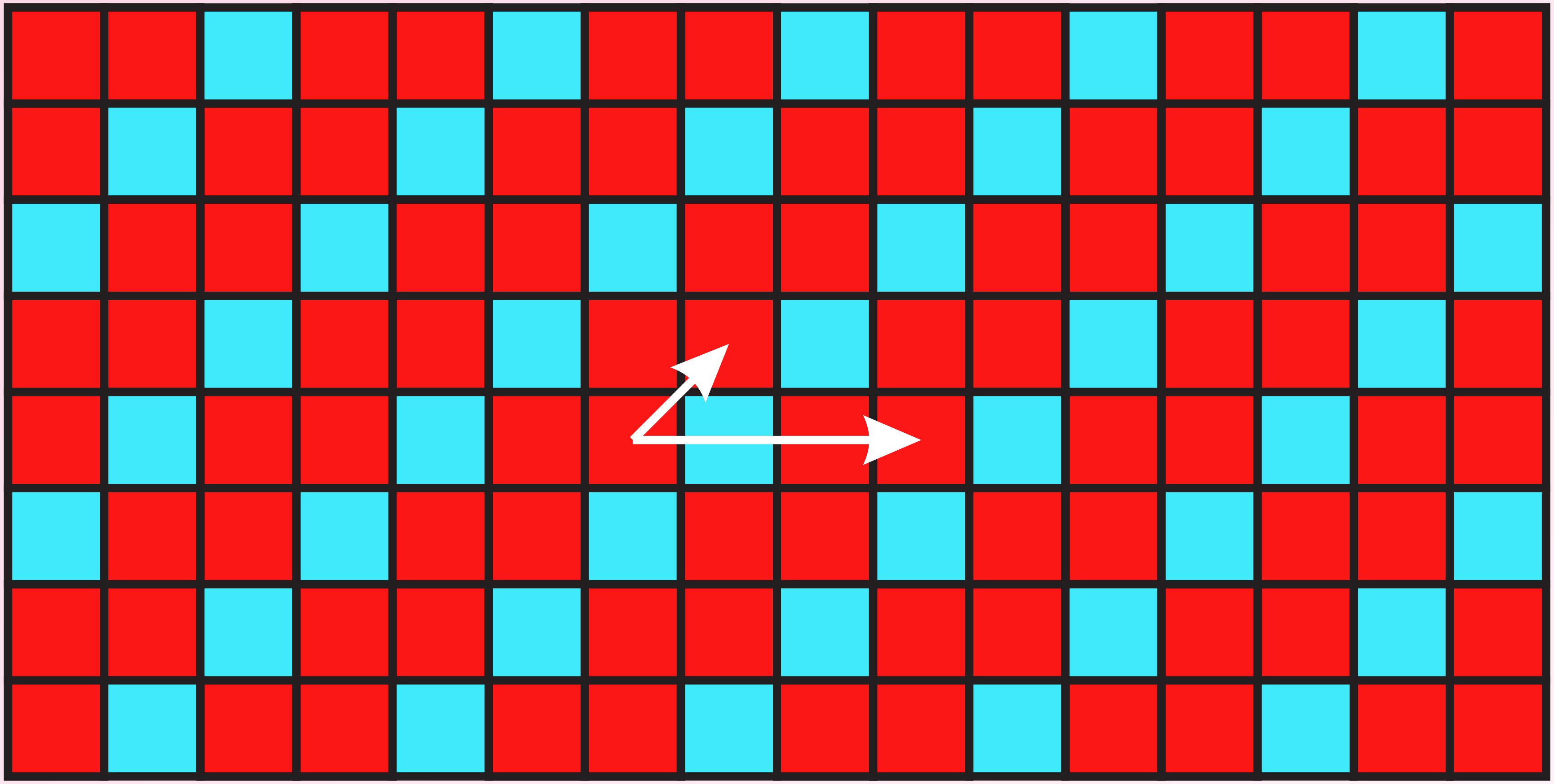


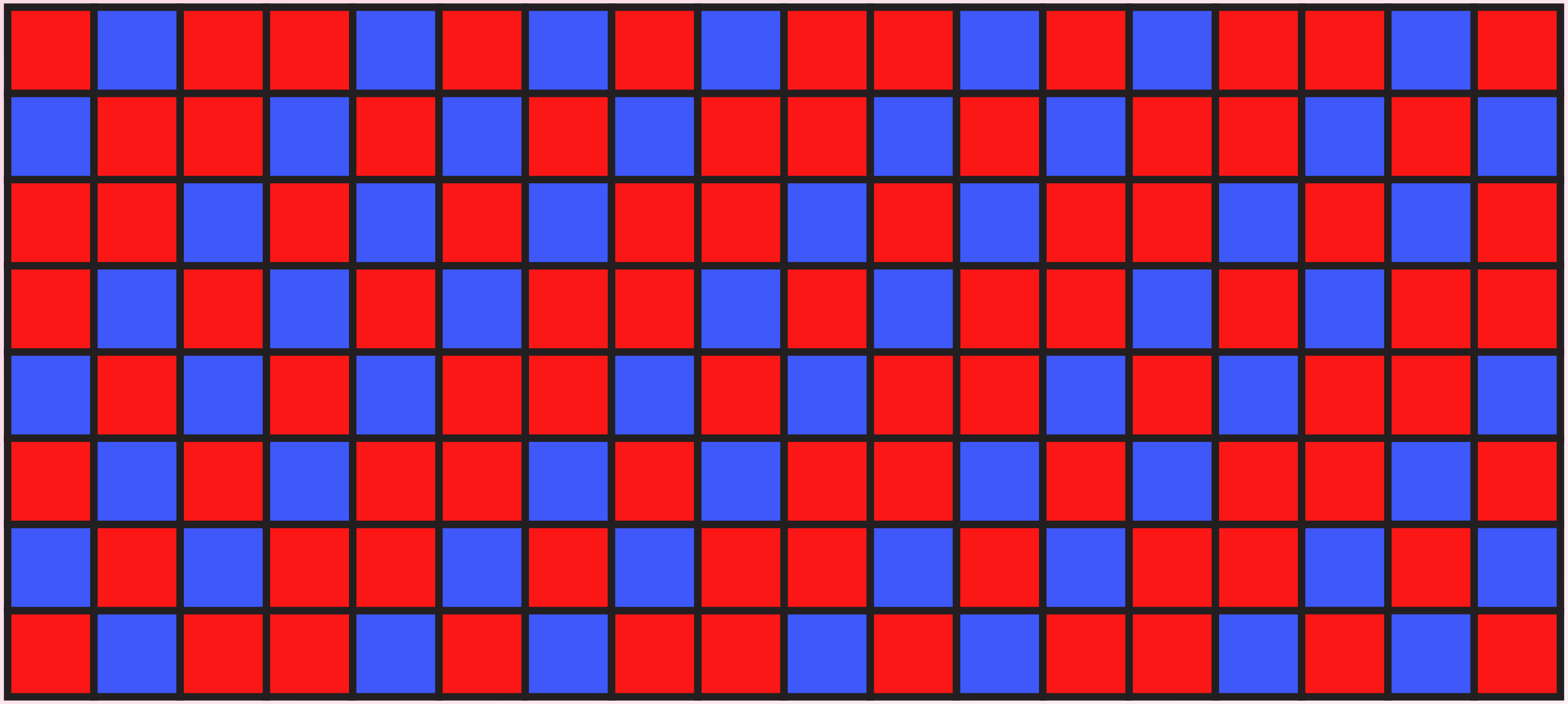
A d-dimensional **configuration** over set A is an element of $A^{\mathbb{Z}^d}$.

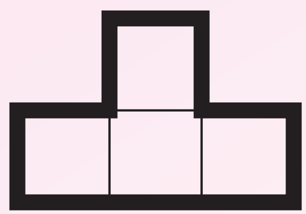


Configuration is **periodic** if it is invariant under a non-zero translation.

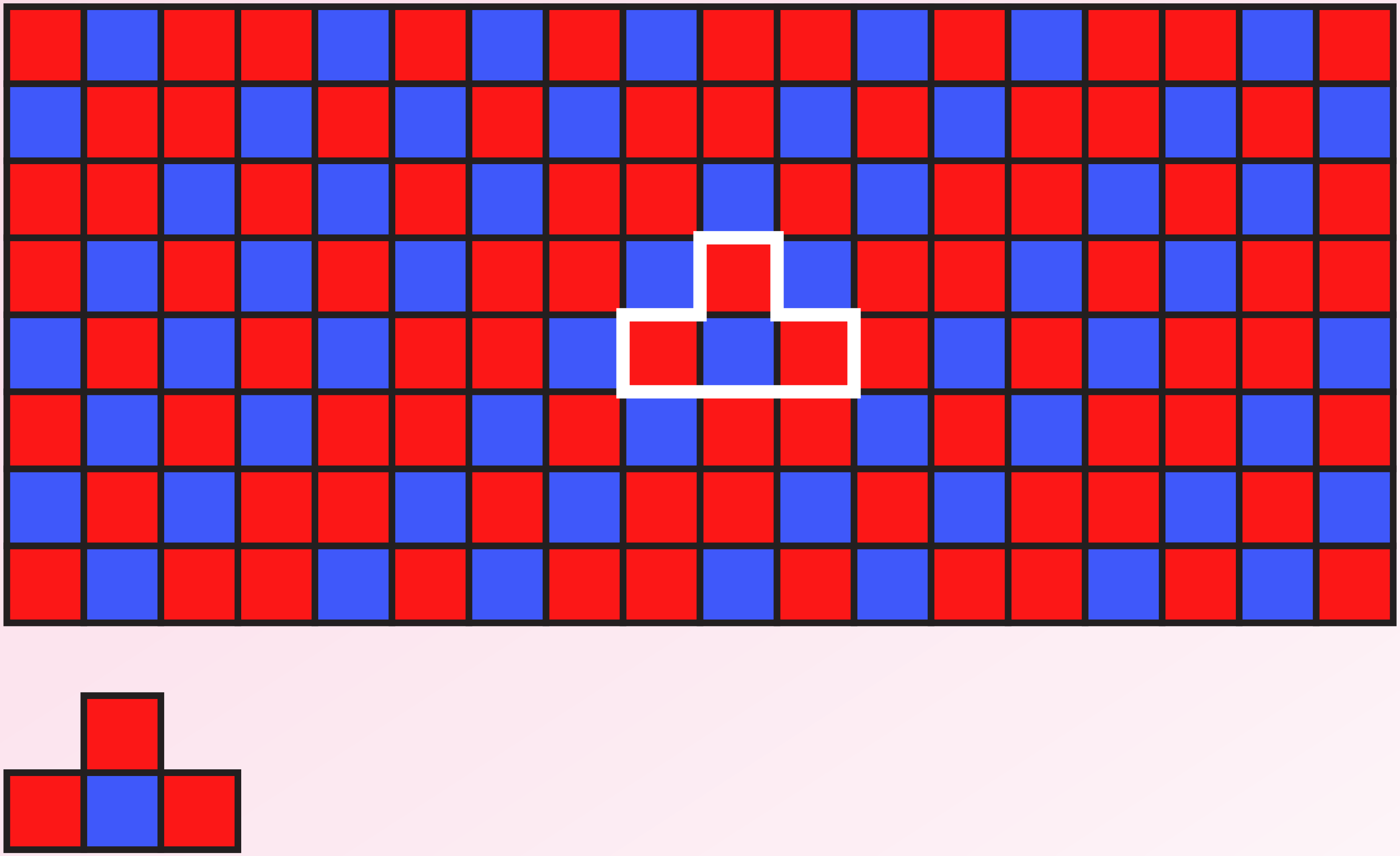


And a 2D configuration is **doubly periodic** if it is periodic in two directions (i.e. has horizontal and vertical periods).





For finite **observation window** $D \subseteq \mathbb{Z}^d$ we call the patterns of shape D that appear in configuration c the **D-patterns** of c.



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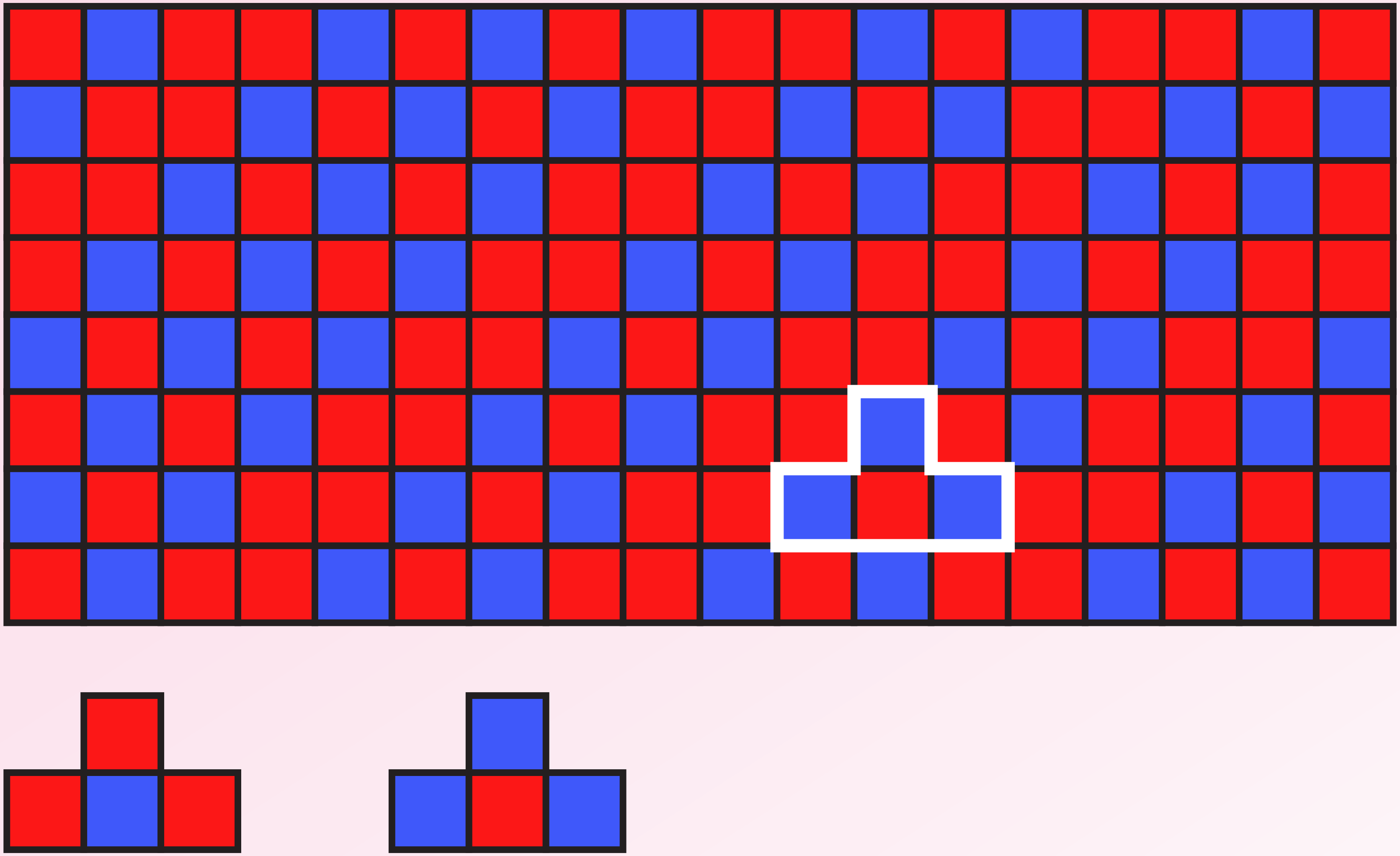

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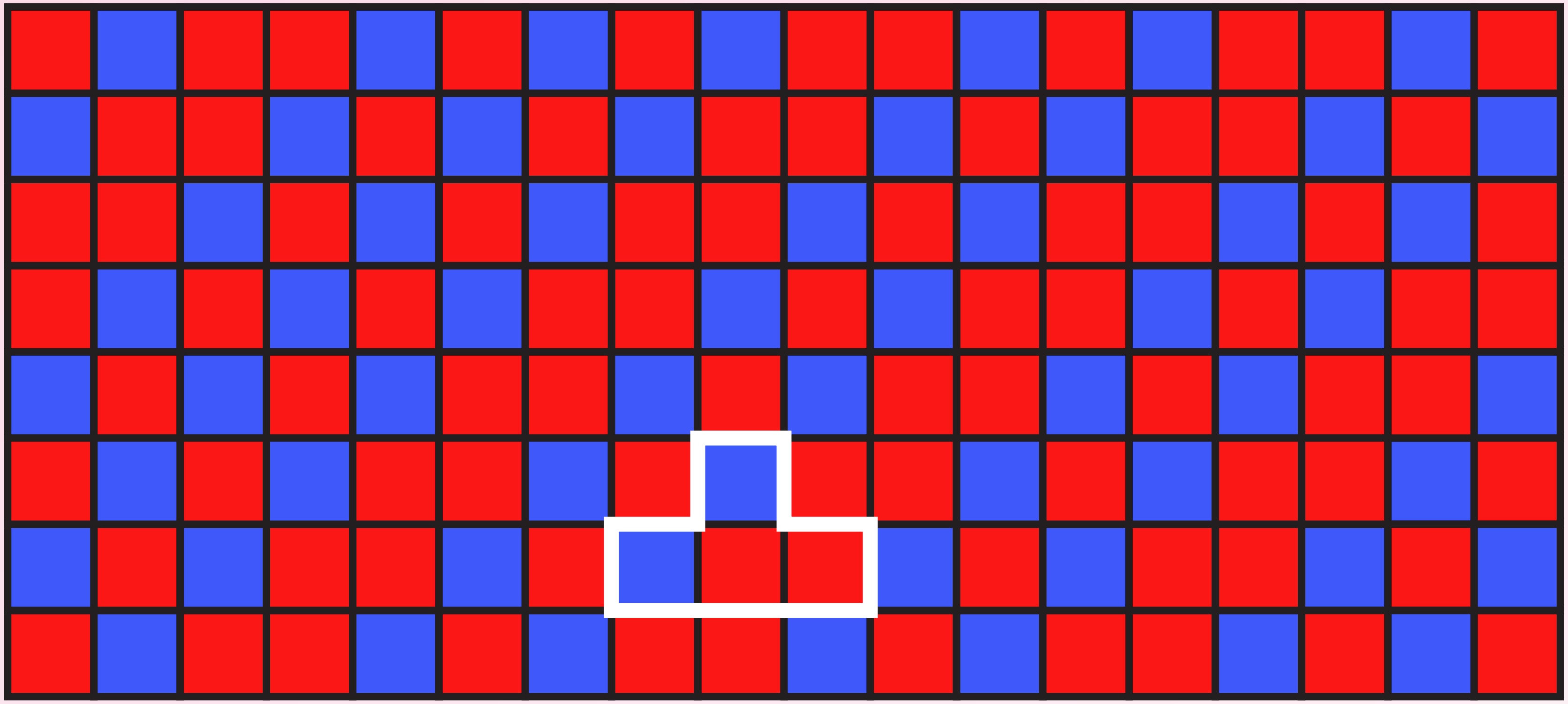

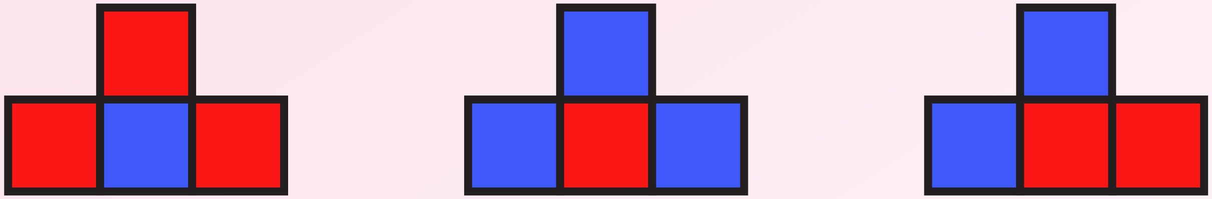

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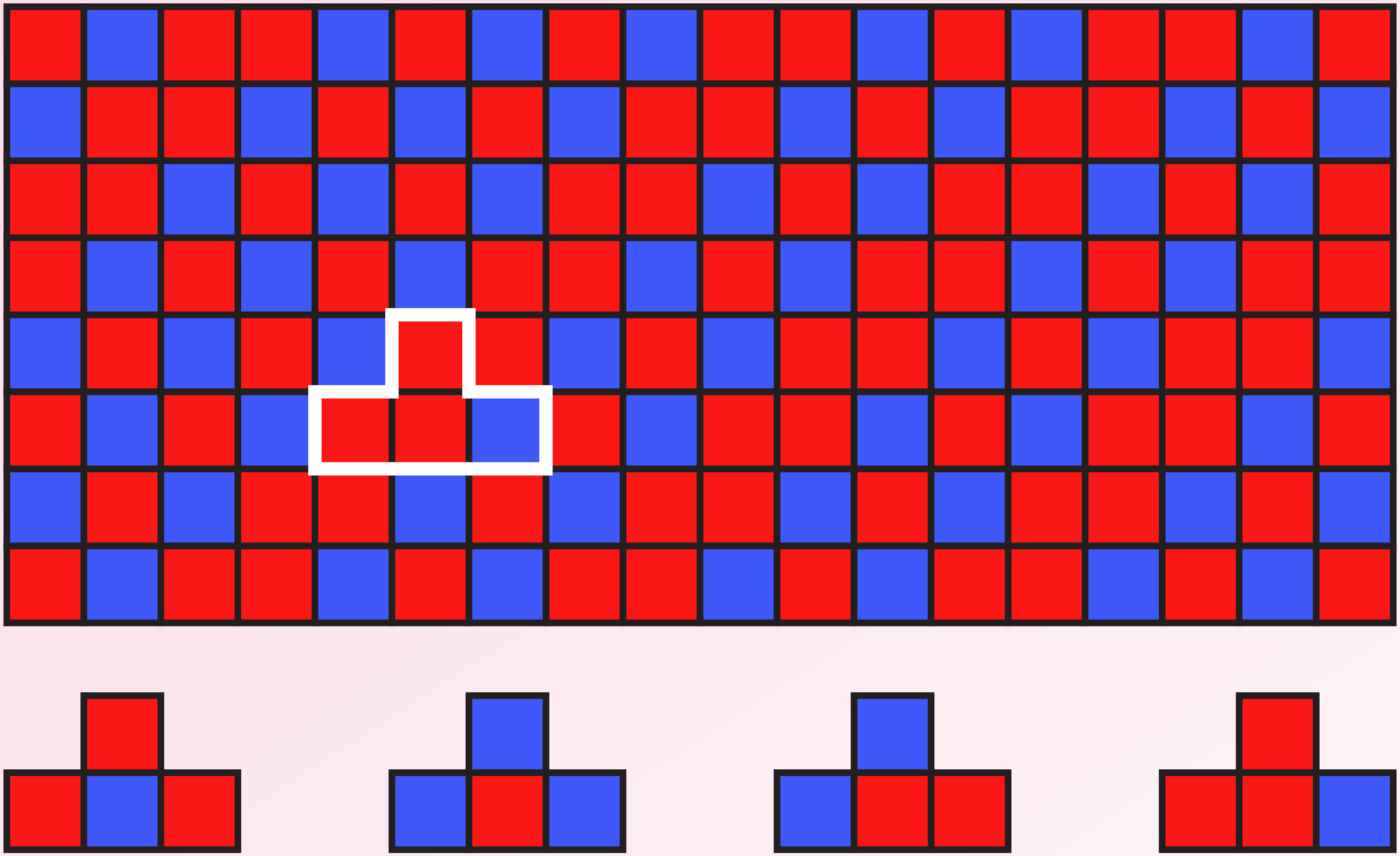

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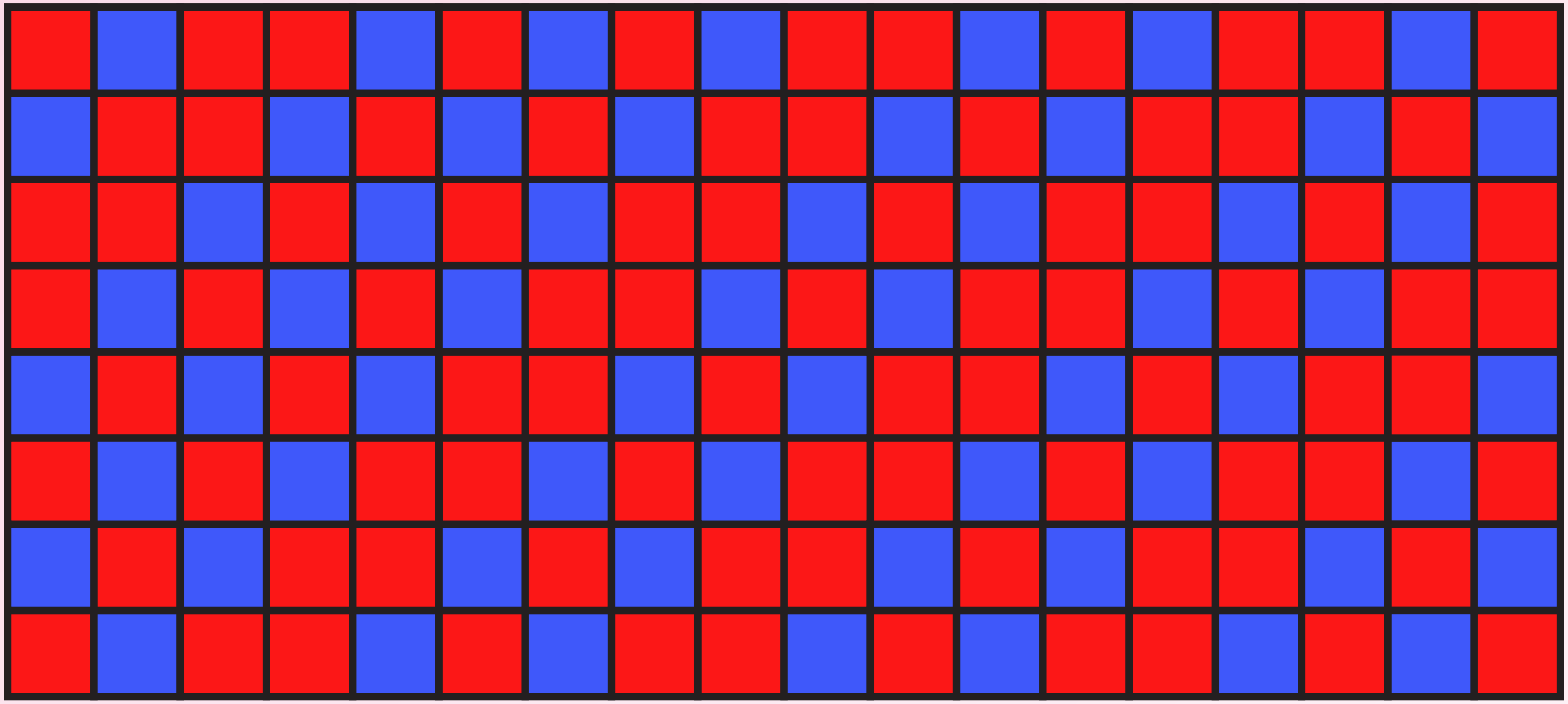
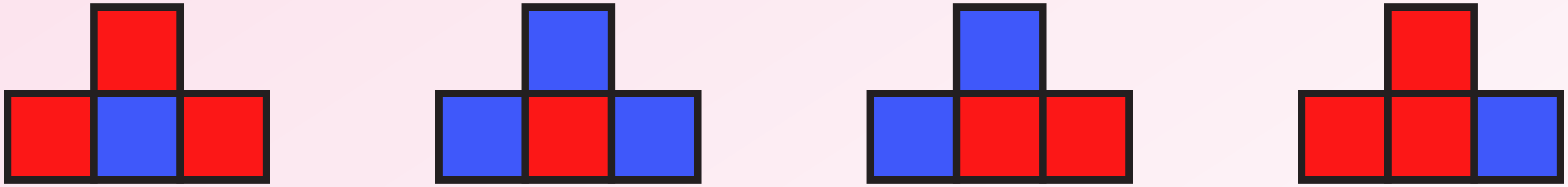




The number of D-patterns is the pattern **complexity**

$$P(c, D) = \#\{\tau(c)_D \mid \tau \text{ is a translation } \}.$$

Local vs. global order

When does small pattern complexity imply periodicity ? Can a small pattern set prevent periodicity ?

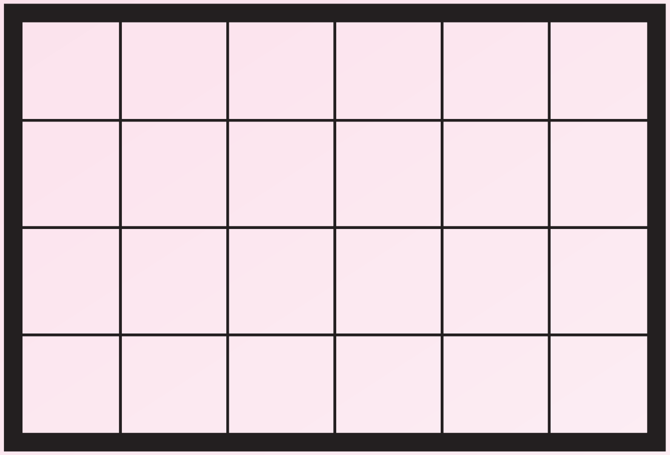
We are interested in the case when

$$P(c, D) \leq |D|$$

for some finite D.

Open problem 1: Nivat's conjecture

Consider $d = 2$ and rectangular D.



Conjecture (Nivat 1997) If $P(c, D) \leq |D|$ for some rectangle D then c is periodic.

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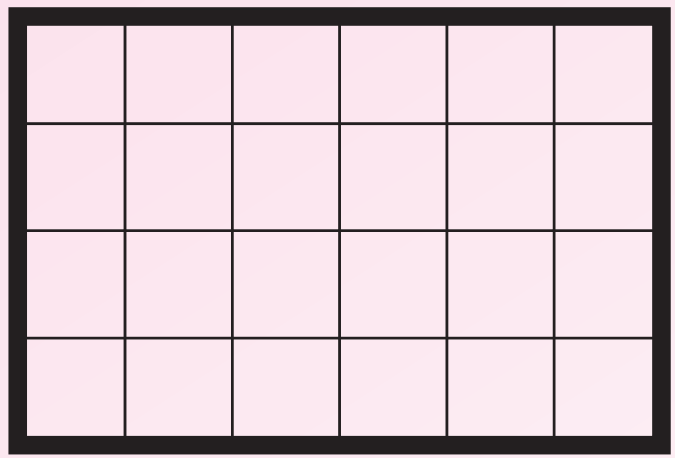


Conjecture (Nivat 1997) If $P(c, D) \leq |D|$ for some rectangle D then c is periodic.

This would extend the one-dimensional case $d = 1$:

Morse-Hedlund theorem: Let $c \in A^{\mathbb{Z}}$ and $n \in \mathbb{N}$. If c has at most n distinct subwords of length n then c is periodic.

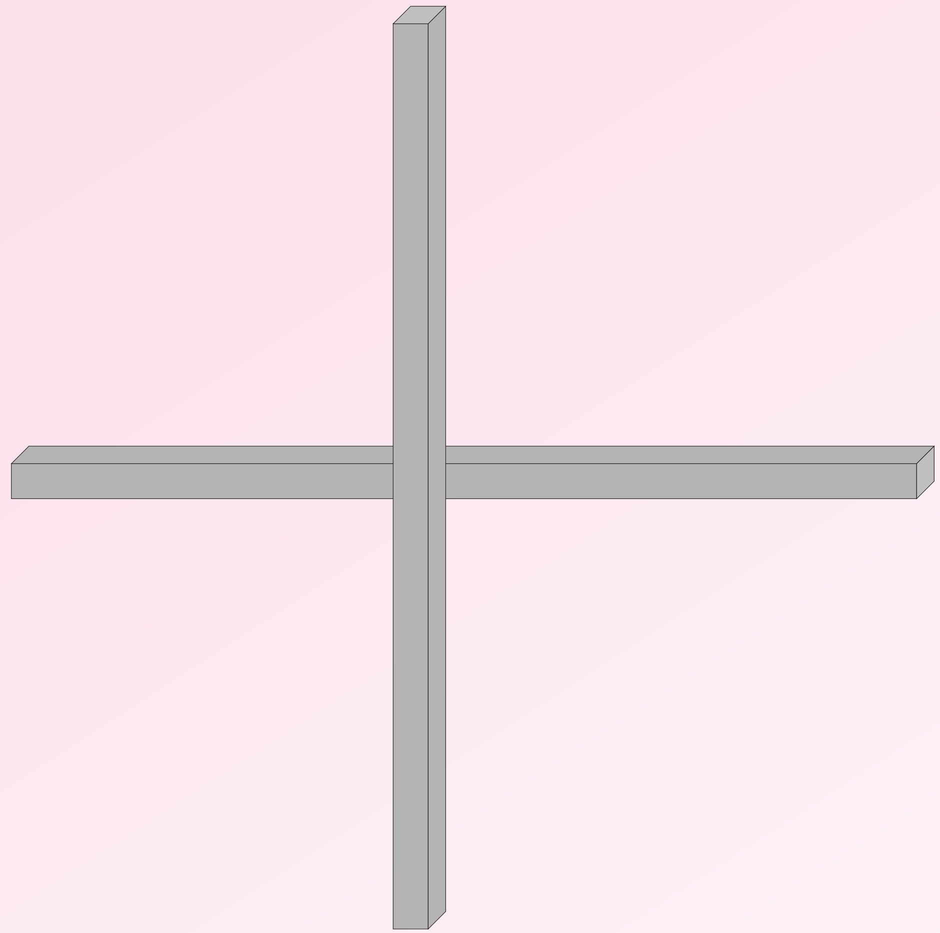
Best known bound in 2D:

Theorem (Cyr, Kra): If $P(c, D) \leq \frac{1}{2}|D|$ for some rectangle D then c is periodic.

Also narrow rectangles have been taken care of:

Theorem (Cyr, Kra): If D is a rectangle of height at most 3 and $P(c, D) \leq |D|$ then c is periodic.

In 3D and higher dimensional cases the conjecture is false



Non-periodic c

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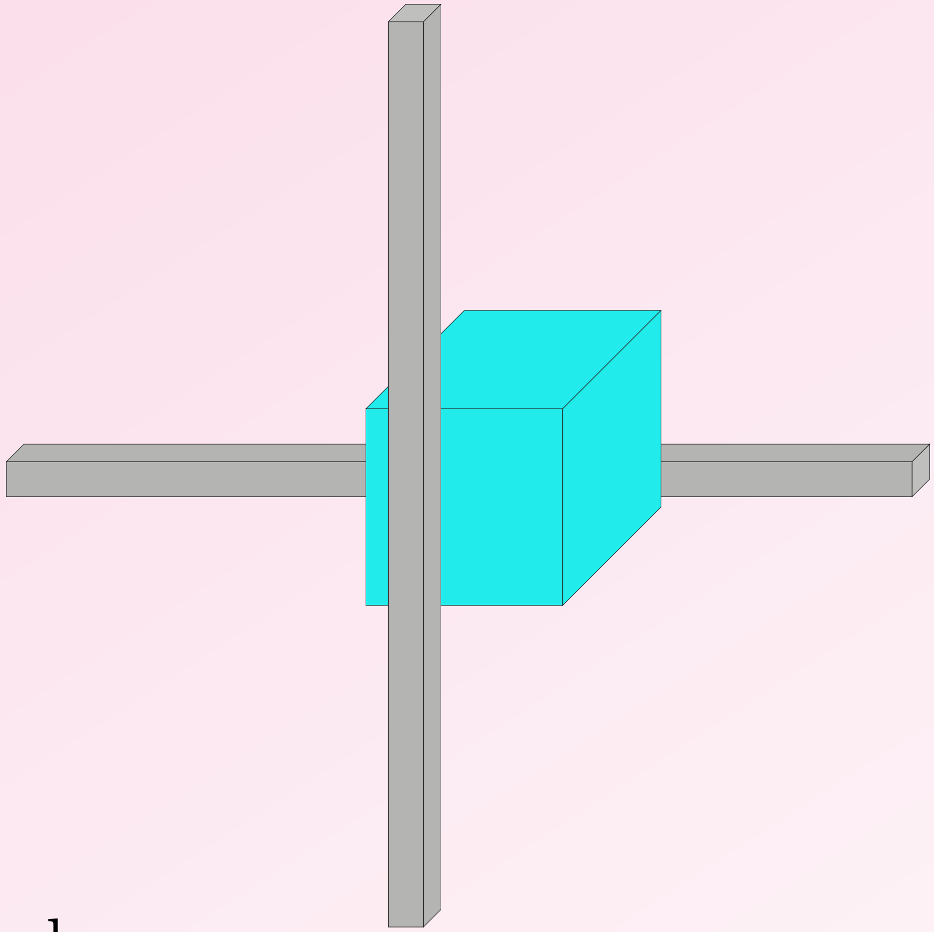

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D is $n \times n \times n$ cube

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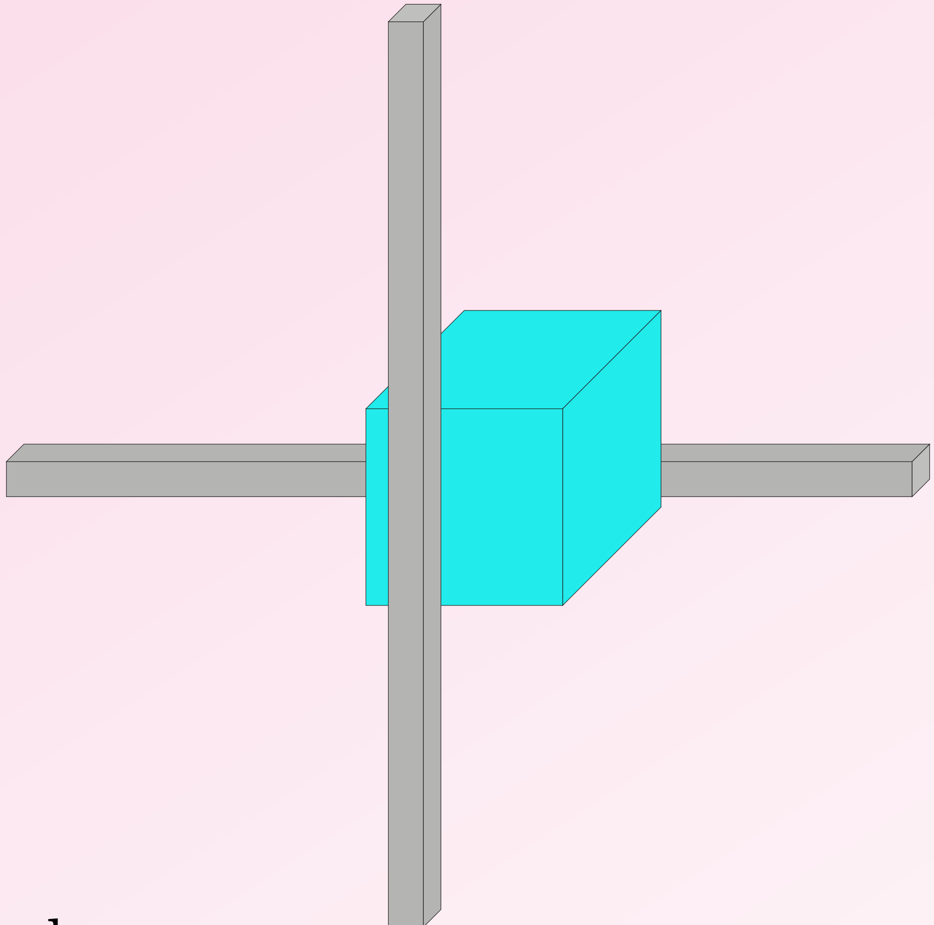

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$P(c, D) = 1 + \ldots$

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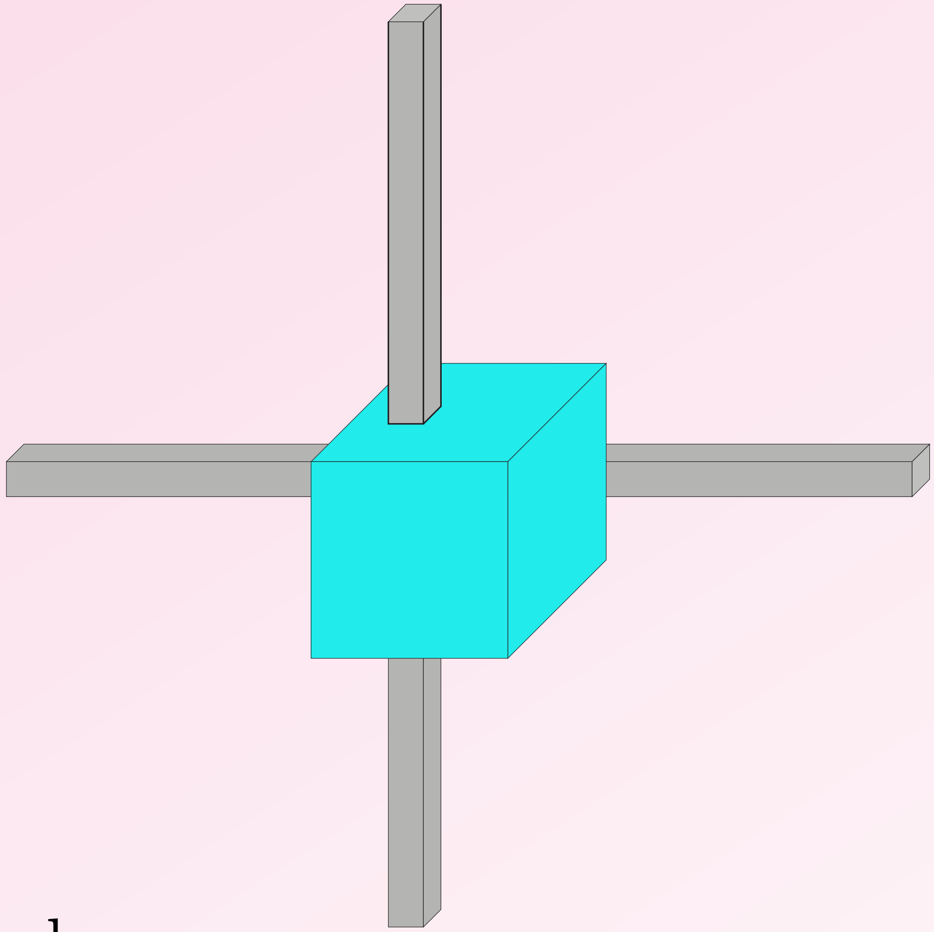

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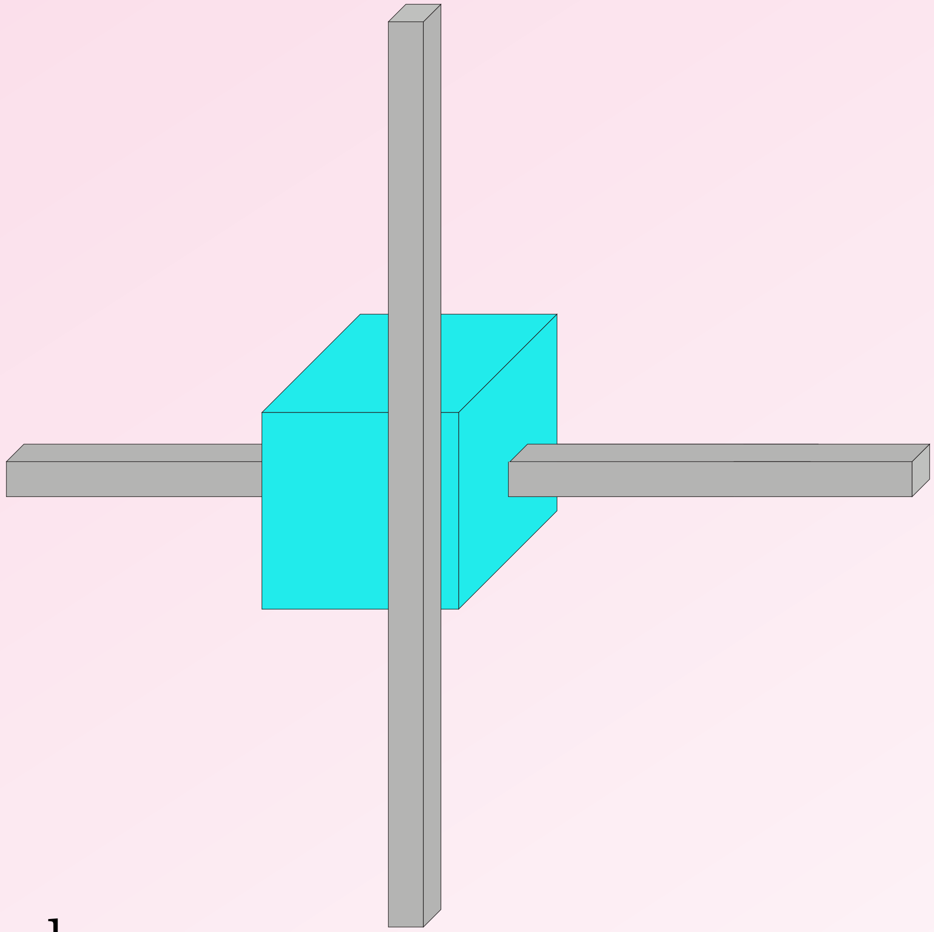

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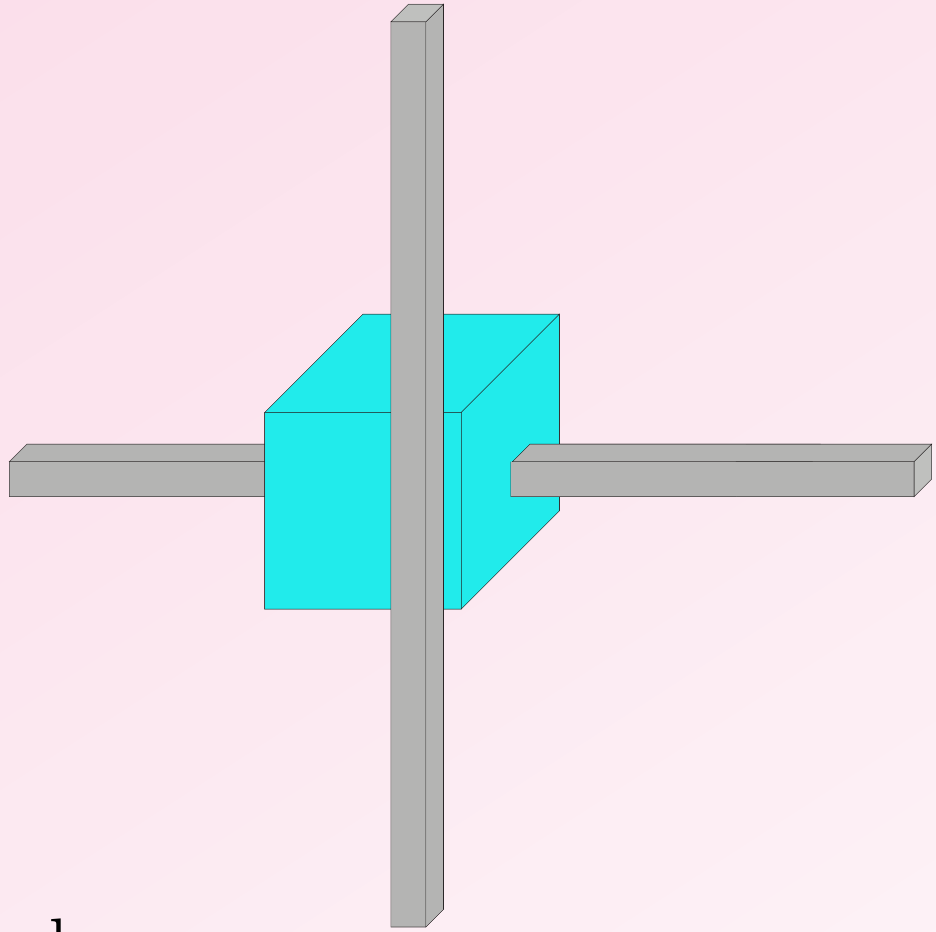

Non-periodic c

D is $n \times n \times n$ cube

$P(c, D) = 1 + n^2 + n^2 < n^3 = |D|$ for large n.

We can prove an asymptotic version in 2D:

Theorem (Kari, Szabados): If $P(c, D) \leq |D|$ for infinitely many different size rectangles D then c is periodic.

Open problem 2: Periodic tiling problem

Let $T \subseteq \mathbb{Z}^d$ be finite, and call it a **tile**. A **tiling** is any $C \subseteq \mathbb{Z}^d$ such that

$$C \oplus T = \mathbb{Z}^d.$$

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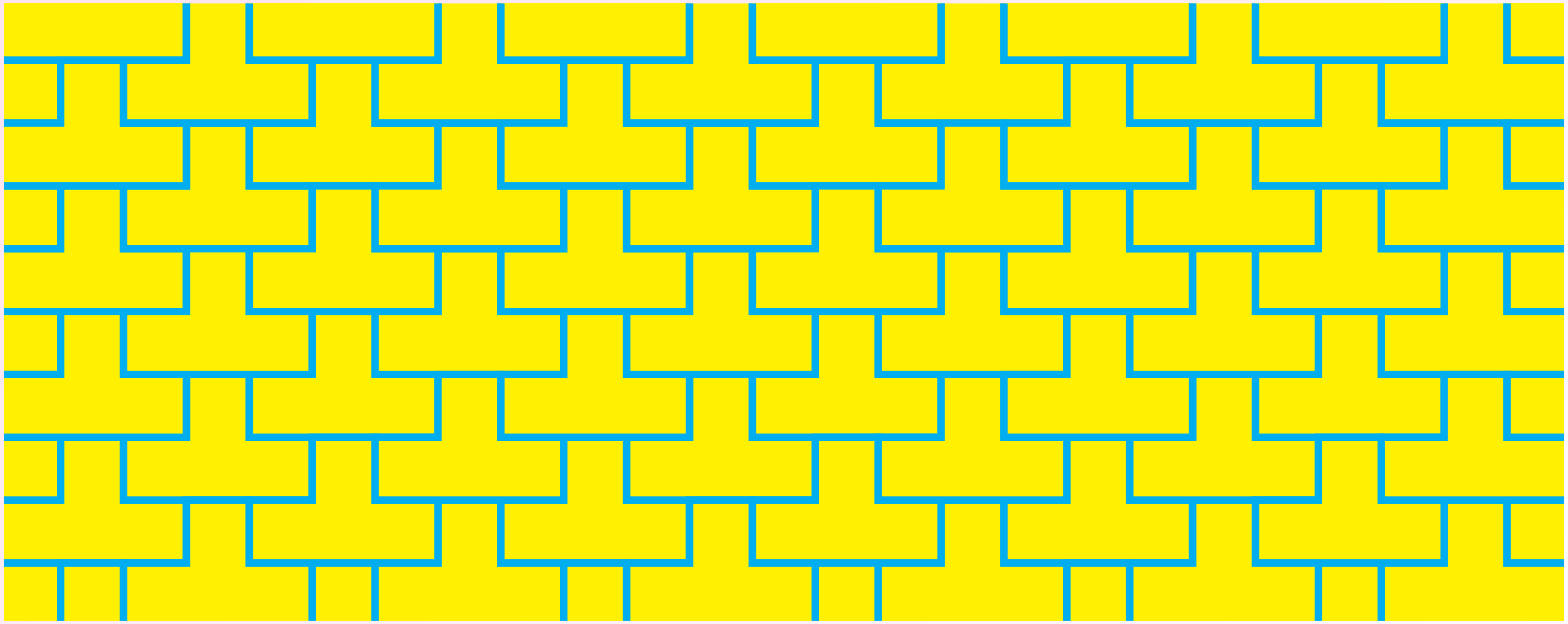
Graphical interpretation: C gives the positions where copies of T are place to cover $\mathbb{Z}^d$ without gaps or overlaps.



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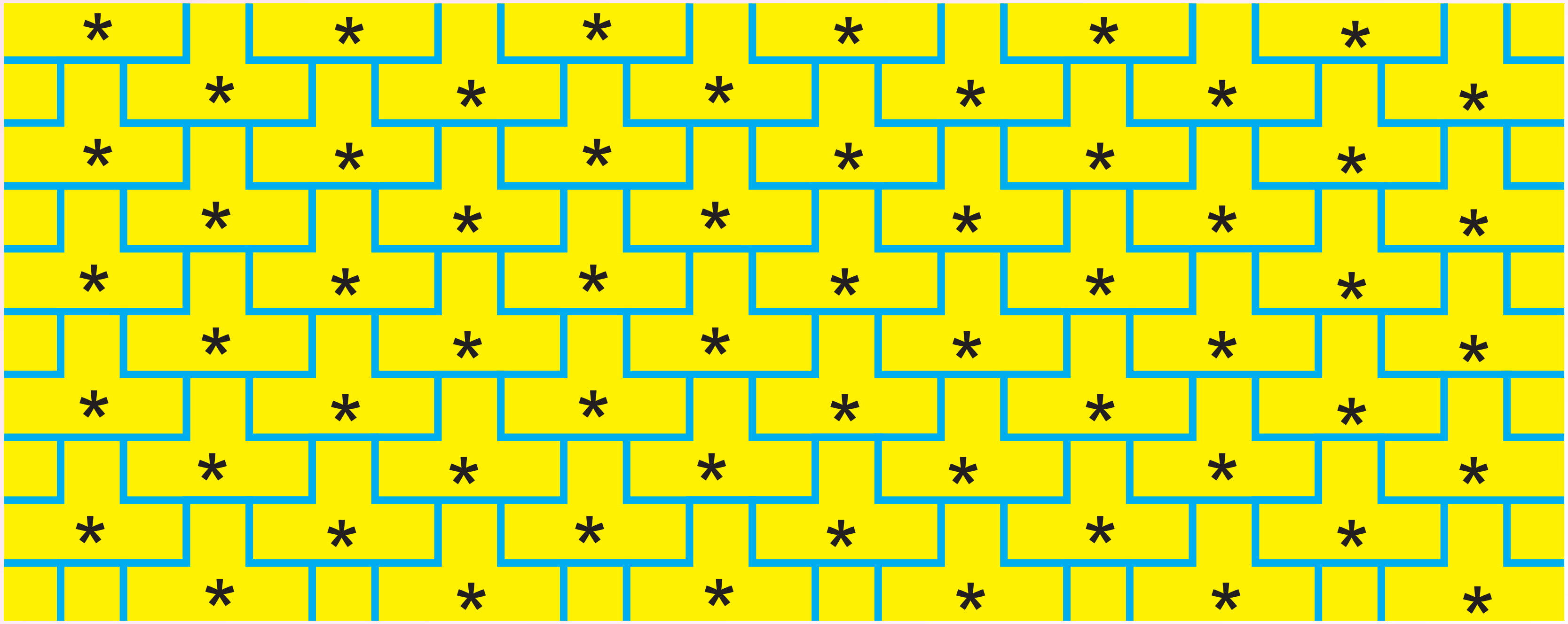

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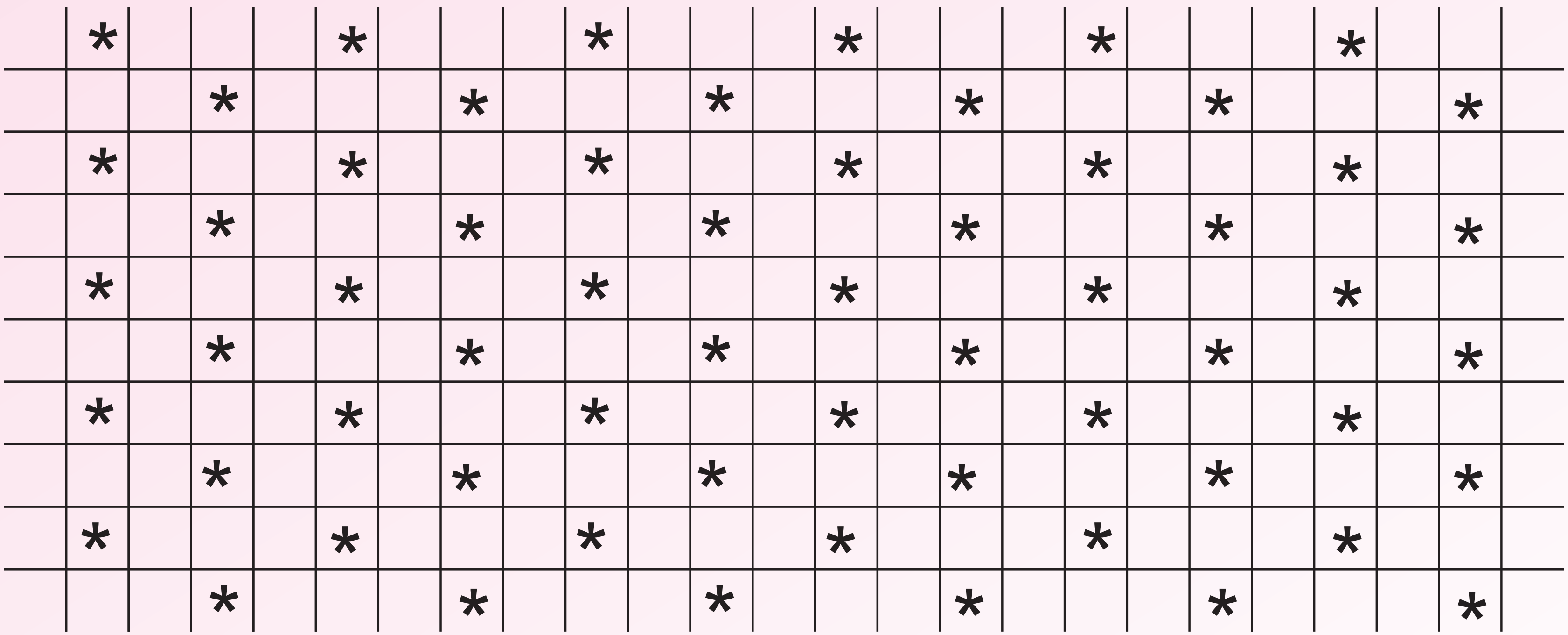
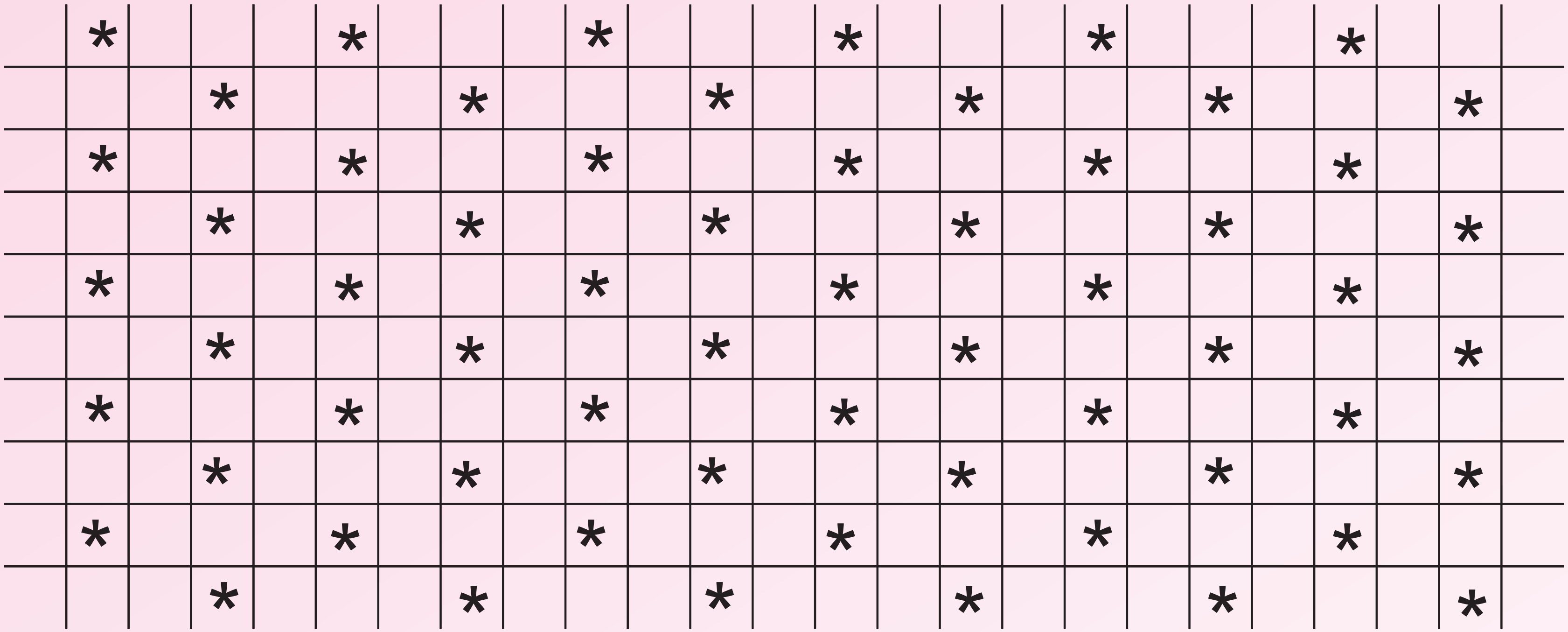
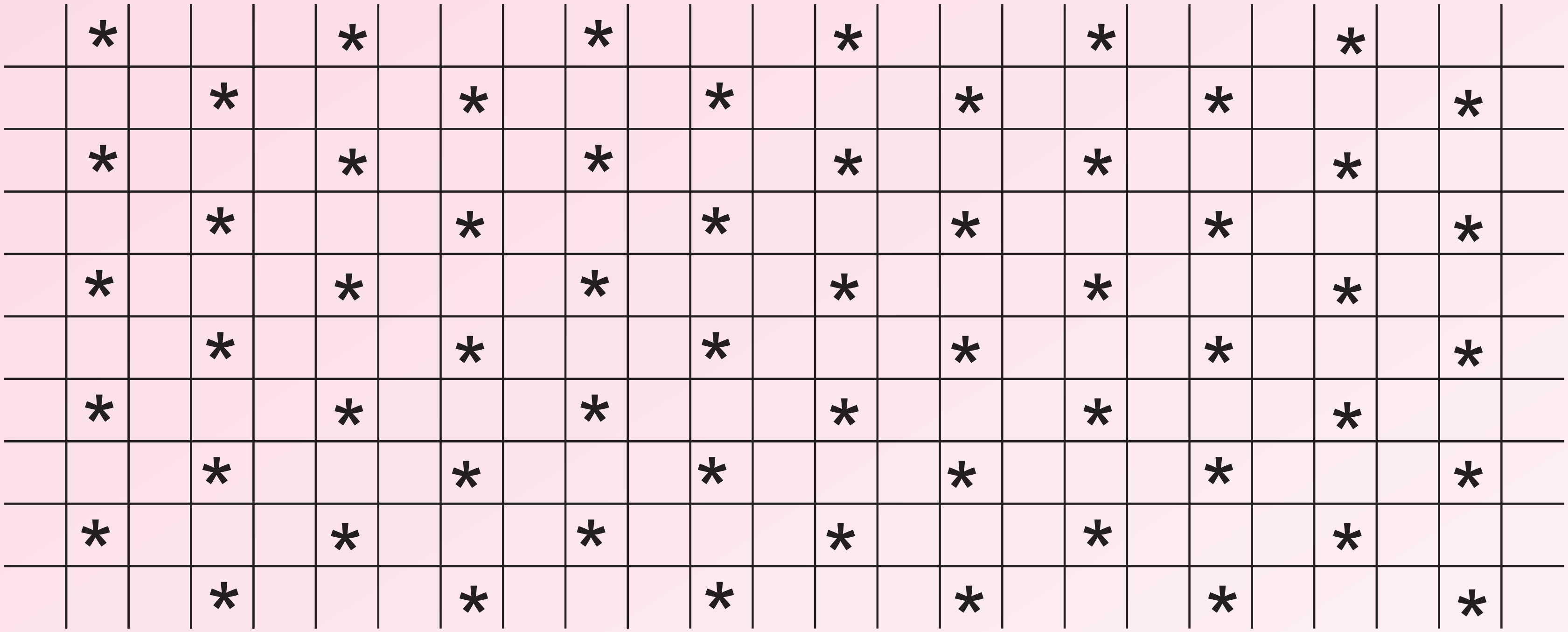
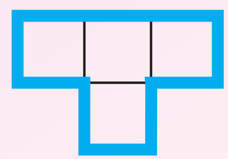




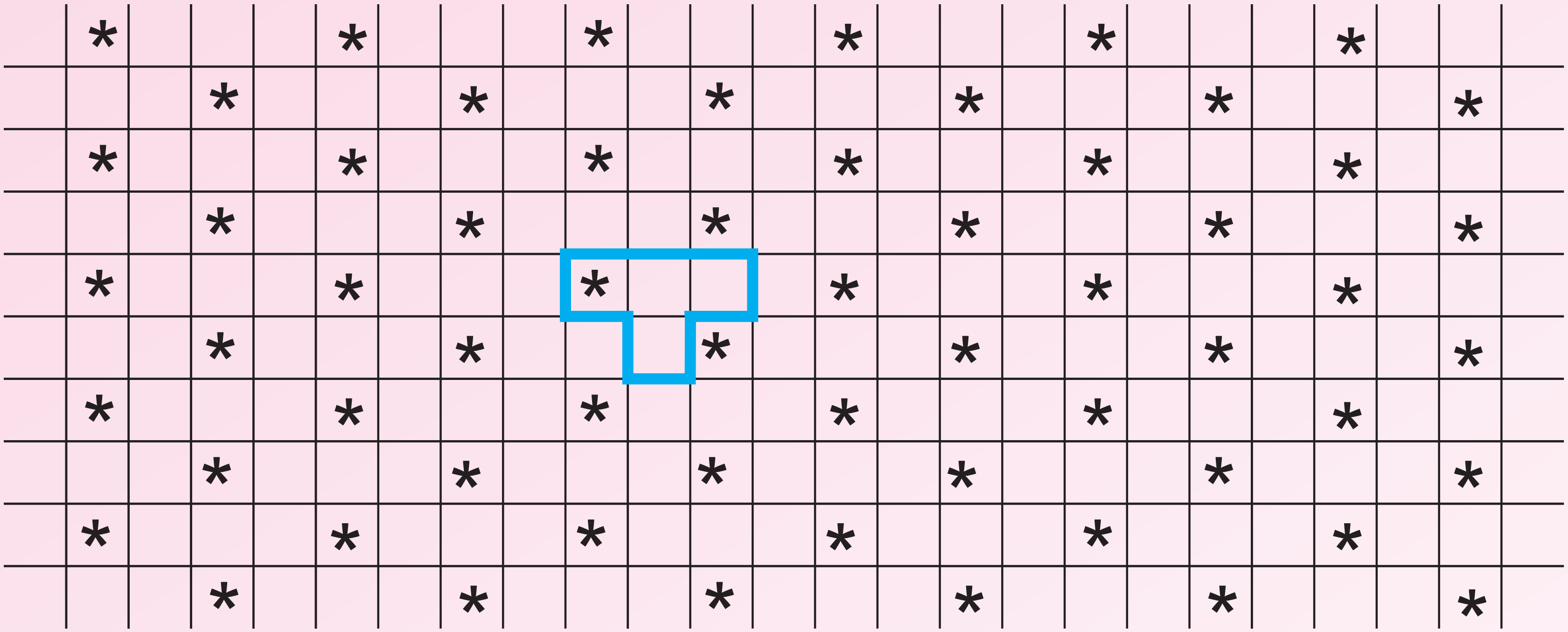
Interpret C as the binary configuration c with

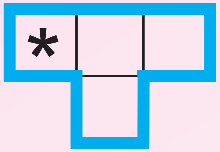
$$c(i) = * \Longleftrightarrow i \in C.$$





$(-T)$-patterns of c contain exactly one symbol $*$.





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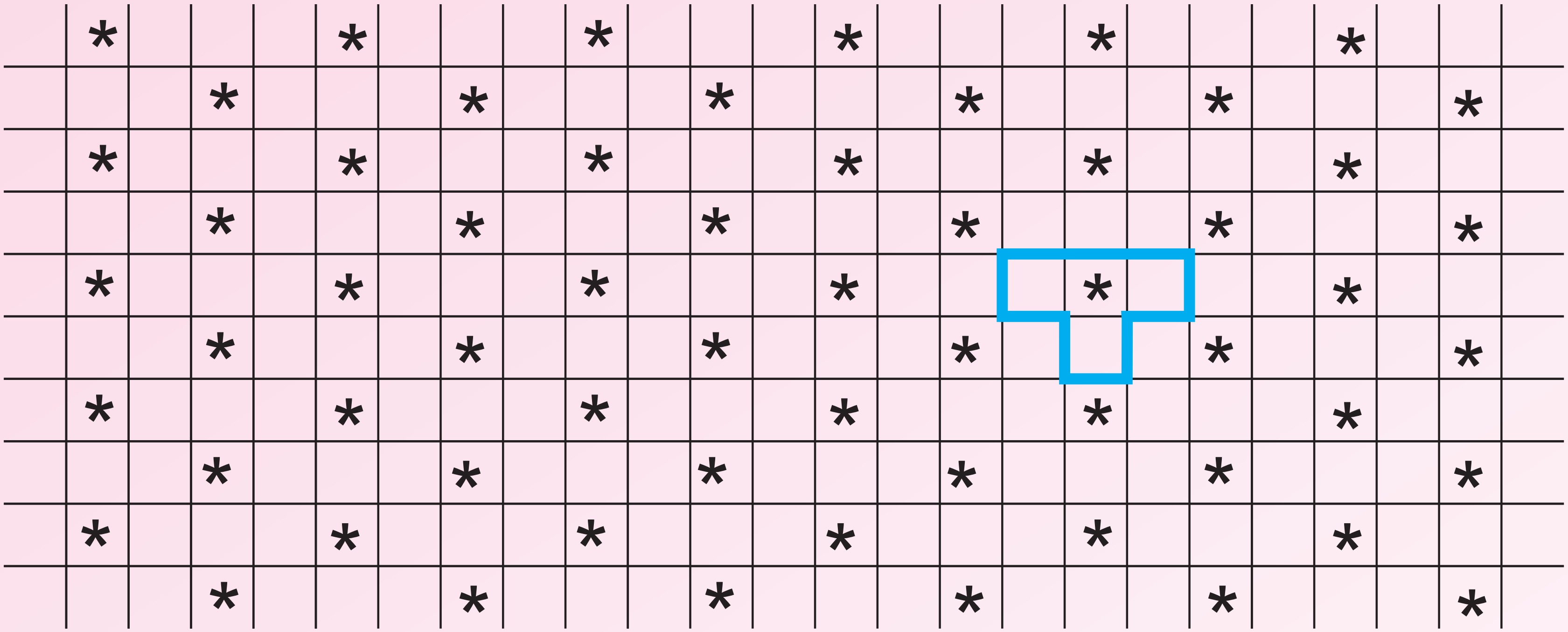

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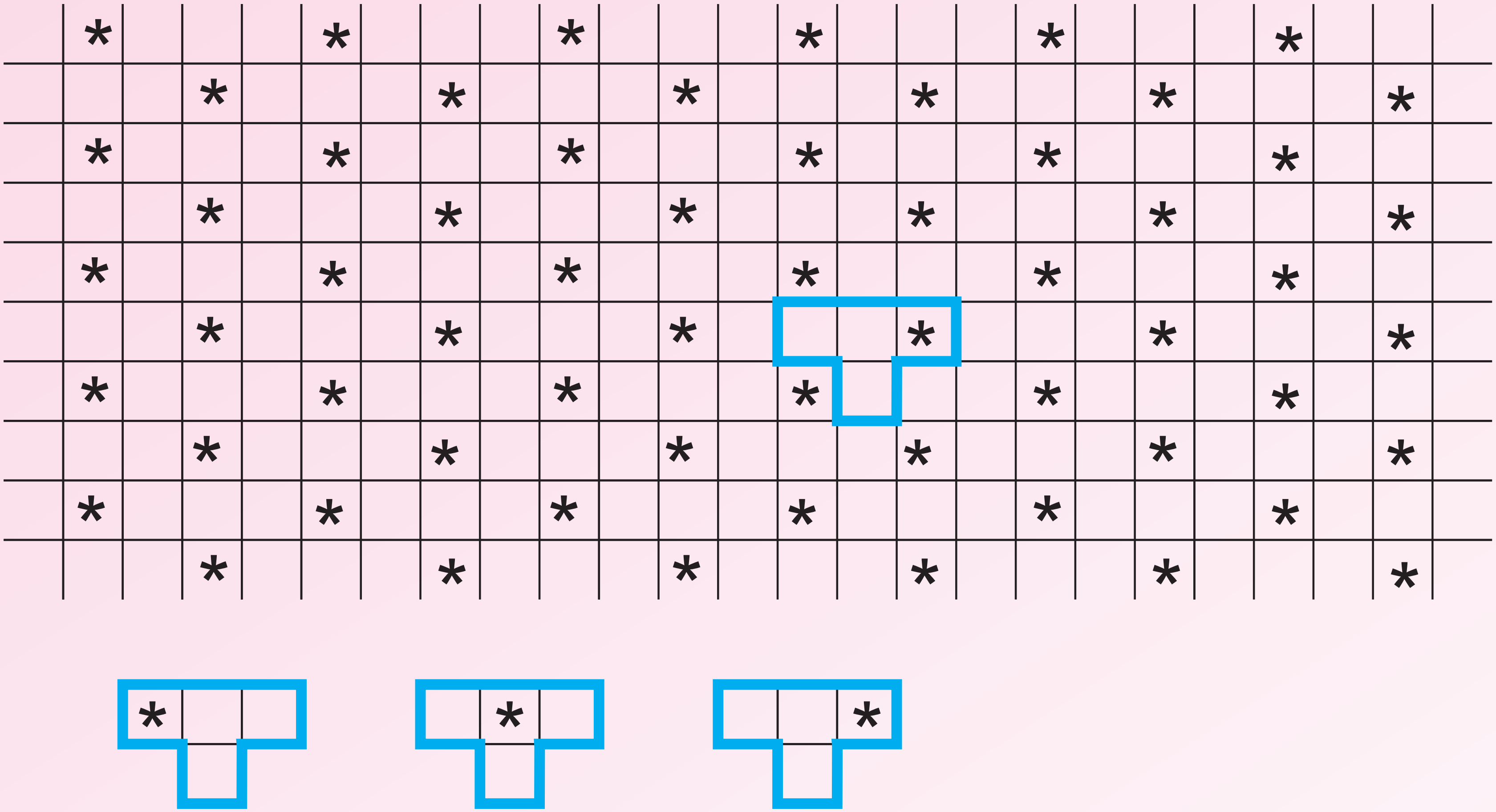

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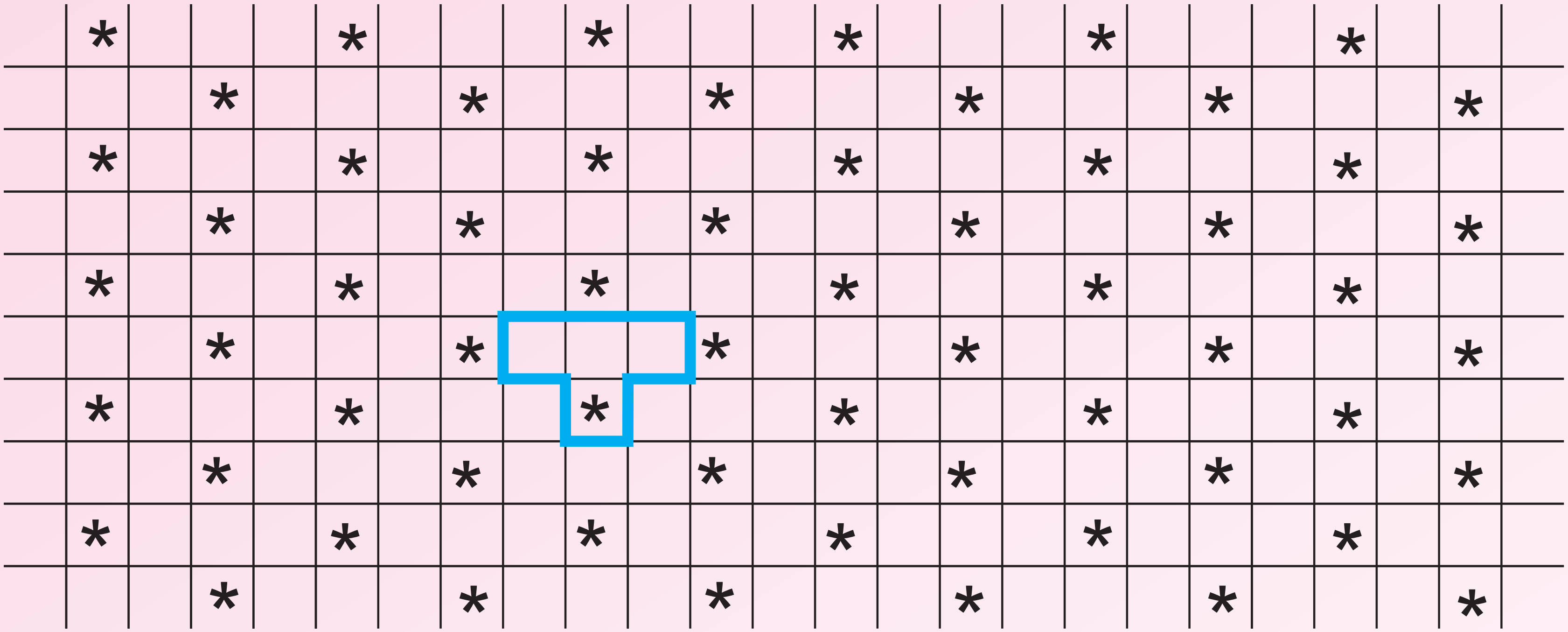

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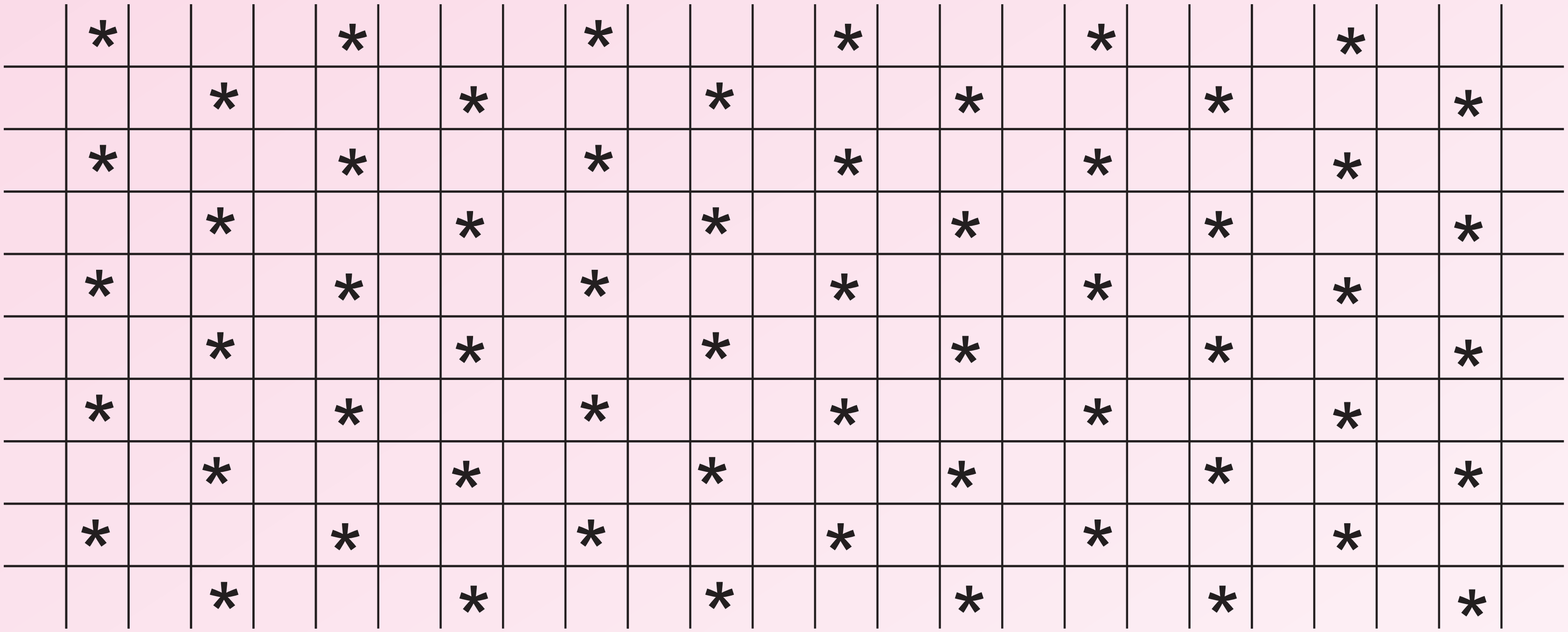
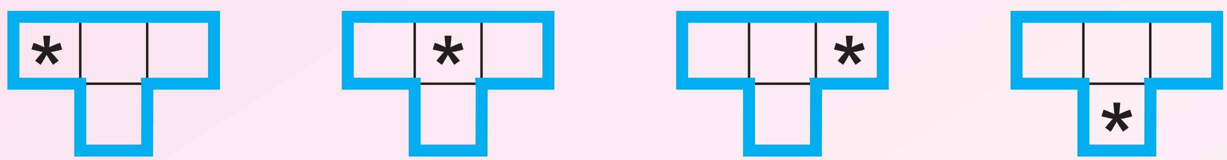
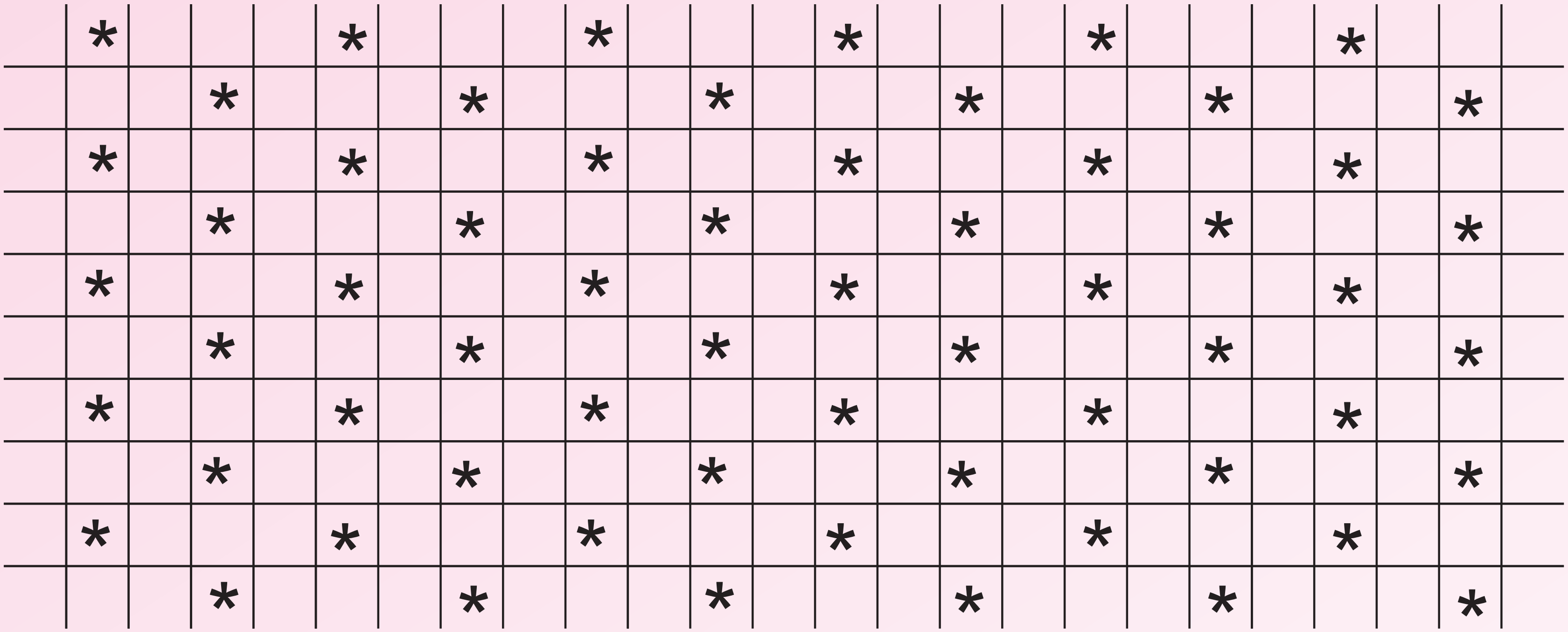




$(-T)$-patterns of c contain exactly one symbol $*$.

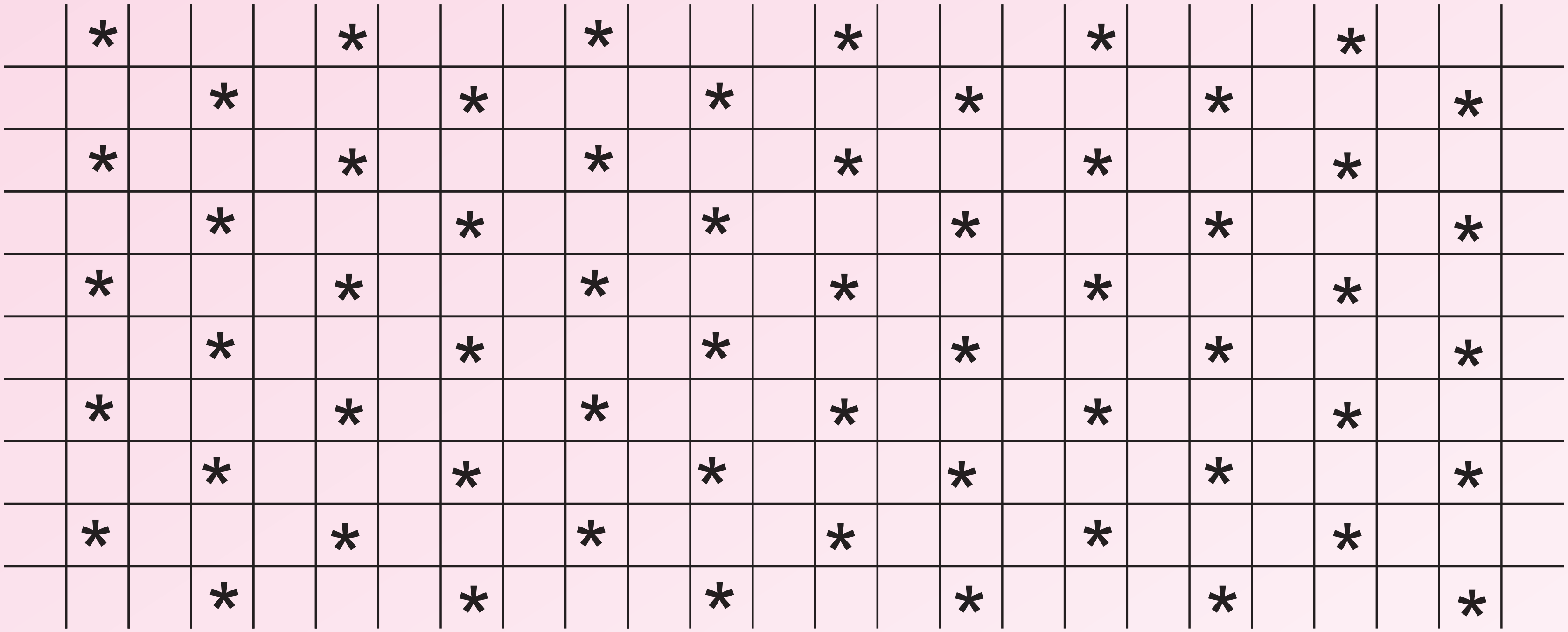
$$P(c, -T) = |T|$$



$(-T)$-patterns of c contain exactly one symbol $*$.

$$P(c, -T) = |T|$$

(Also $P(c, T) = |T|$ as any tiling for T is also a tiling for $-T$.)



If X is the **SFT of all tilings** by T then

$$P(X,T) = |T|$$

where $P(X,T)$ is defined in the obvious way (=number of distinct T-patterns found in X).

Periodic tiling problem (Lagarias and Wang 1996): If T admits a tiling C, does it necessarily admit a periodic tiling ?

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Known results:

- Yes if $|T|$ is a prime number (Szegedy 1998).
- Yes in 2D if T is 4-connected (Beauquier and Nivat 1991).

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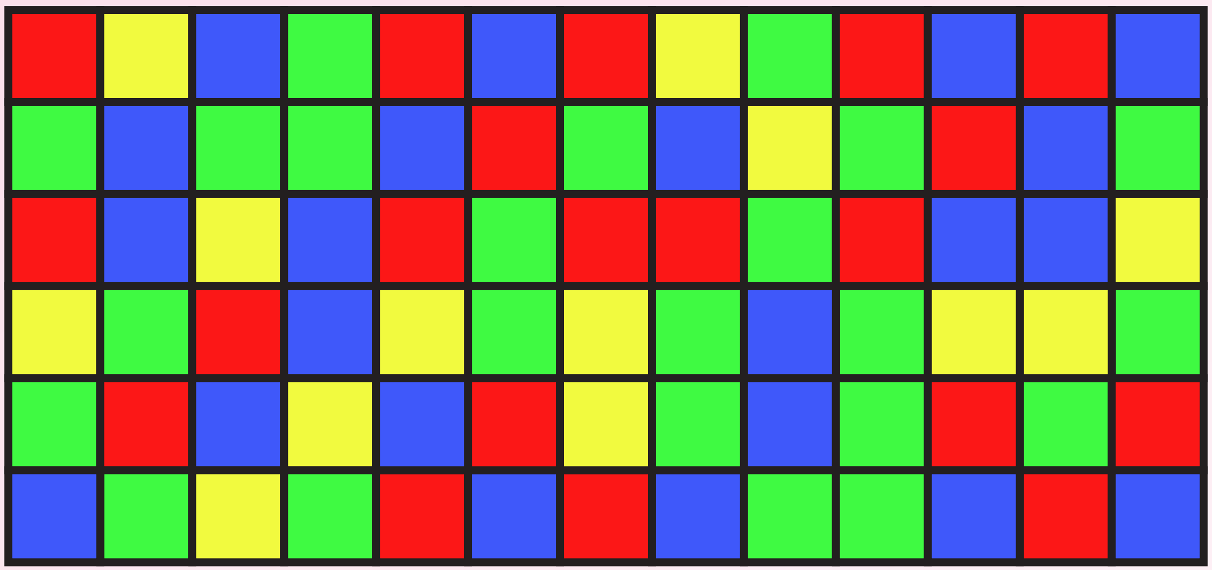
Both the **Nivat's conjecture** and the **Periodic tiling problem** concern periodicity under complexity constraint $P(c, T) \leq |T|$.

We study configurations using algebra, so we first replace symbols by integers:



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1	2	3	4	1	3	1	2	4	1	3	1	3
4	3	4	4	3	1	4	3	2	4	1	3	4
1	3	2	3	1	4	1	1	4	1	3	3	2
2	4	1	3	2	4	2	4	3	4	2	2	4
4	1	3	2	3	1	2	4	3	4	1	4	1
3	4	2	4	1	3	1	3	4	4	3	1	3

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2	4	1	3	2	4	2	4	3	4	2	2	4
4	1	3	2	3	1	2	4	3	4	1	4	1
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D-patterns are viewed as $|D|$-dimensional numerical vectors.

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D-patterns are viewed as $|D|$-dimensional numerical vectors.

If $P(c, D) < |D|$ then there is an (integer) vector orthogonal to all D-patterns of c. **Filtering** c by this vector produces zero-configuration.

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1	2	3	4	1	3	1	2	4	1	3	1	3
4	3	4	4	3	1	4	3	2	4	1	3	4
1	3	2	3	1	4	1	1	4	1	3	3	2
2	4	1	3	2	4	2	4	3	4	2	2	4
4	1	3	2	3	1	2	4	3	4	1	4	1
3	4	2	4	1	3	1	3	4	4	3	1	3

If $P(c, D) \leq |D|$ we can add a suitable rational constant to c so that an orthogonal vector exists.

This is OK: we are free to choose the encoding of symbols as numbers freely.

We study configurations using algebra, so we first replace symbols by integers:

1	2	3	4	1	3	1	2	4	1	3	1	3
4	3	4	4	3	1	4	3	2	4	1	3	4
1	3	2	3	1	4	1	1	4	1	3	3	2
2	4	1	3	2	4	2	4	3	4	2	2	4
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3	4	2	4	1	3	1	3	4	4	3	1	3

Conclusion: If $P(c, D) \leq |D|$ then symbols can be represented as integers in such a way that a non-trivial integer filter annihilates c.

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1	2	3	4	1	3	1	2	4	1	3	1	3
4	3	4	4	3	1	4	3	2	4	1	3	4
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To use algebraic geometry, we next represent c as a power series (negative exponents included):

$$c \longleftrightarrow \sum_{(i_1,\ldots,i_d)\in\mathbb{Z}^d} c(i_1,\ldots,i_d)x_1^{i_1}\ldots x_d^{i_d}$$

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Notations:

- $X = (x_1, \ldots, x_d)$
- For $I = (i_1, \ldots, i_d) \in \mathbb{Z}^d$ we denote by

  $$X^I = x_1^{i_1} \ldots x_d^{i_d}$$

  the monomial that represents cell I.
- For $n \in \mathbb{Z}$ we use $X^n = X^{(n,\ldots,n)} = x_1^n \ldots x_d^n$.

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The configuration is now a power series $c(X)$ that is

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More notations:

Let R be a ring. (Usually $\mathbb{Z}$ or $\mathbb{C}$.)

- We denote $Z^I = z_1^{i_1} \dots z_d^{i_d}$ for any

$$\begin{aligned} Z &= (z_1, \dots, z_d) \in R^d, \\ I &= (i_1, \dots, i_d) \in \mathbb{Z}^d. \end{aligned}$$

- $R[X]$ is the set of polynomials over ring R.
- $R[X^{\pm 1}]$ is the set of Laurent polynomials over ring R.
- $R[[X^{\pm 1}]]$ is the set of power series (negative exponents included) over ring R.

A filter is a Laurent polynomial $f(X)$, and filtering corresponds to multiplying a power series with $f(X)$.

We say that $f(X)$ **annihilates** $c(X)$ if $f(X)c(X) = 0$. Zero polynomial $f(X) = 0$ annihilates every configuration – it is the **trivial annihilator**.

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Our setup is

$$c(X) \in \mathbb{Z}[[X^{\pm 1}]]$$

that has non-trivial annihilators

$$f(X) \in \mathbb{Z}[X^{\pm 1}].$$

Monomial X^I represents the **translation** by vector $I \in \mathbb{Z}^d$. If $f(X)$ annihilates $c(X)$ then so does $X^I \cdot f(X)$.

For any Laurent polynomial $f(X)$ there exists I such that $X^I \cdot f(X)$ is a polynomial.

$\Longrightarrow$ our $c(X)$ has non-trivial **annihilating polynomials.**

Define

$$\mathrm{Ann}(c) = \{f(X) \in \mathbb{C}[X] \mid f(X)c(X) = 0\}.$$

This is the polynomial ideal that contains all annihilators of c.

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Remarks:

- We consider polynomials (not Laurent polynomials!) so that we can directly rely on polynomial algebra.
- We allow complex coefficients because we need algebraicly closed field to apply Hilbert's Nullstellensatz.
- $\text{Ann}(c)$ is indeed an ideal of the polynomial ring $\mathbb{C}[X]$.

In the following we prove that $\mathrm{Ann}(c)$ contains polynomials of "simple form", using

Nullstellensatz (Hilbert): Let $g(X)$ be a polynomial. Suppose that $g(Z) = 0$ for all Z in the **variety**

$$\{ Z \in \mathbb{C}^d \mid \; f(Z) = 0 \text{ for all } f \in \mathrm{Ann}(c) \; \}.$$

Then $g^k \in \mathrm{Ann}(c)$ for some $k \in \mathbb{N}$.

In the following, always, $c(X)$ is a finitary, integral power series and

$$f(X) = \sum_{I \in \mathcal{I}} a_I X^I$$

is its non-trivial integral annihilator polynomial, $a_I \neq 0$ for all $I \in \mathcal{I}$.

Lemma 1: There is an integer M such that $f(X^n) \in \mathrm{Ann}(c)$ for every $n \in \mathbb{N}$ that is relatively prime with M.

Number M only depends on $c(X)$ and the coefficients of $f(X)$. (Same M works for all polynomials with the same coefficients.)

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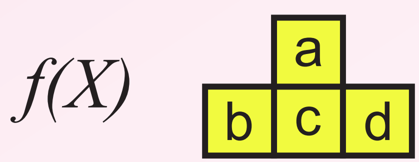

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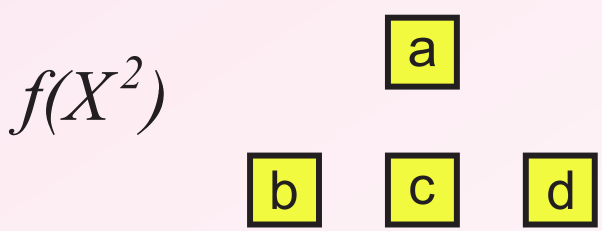

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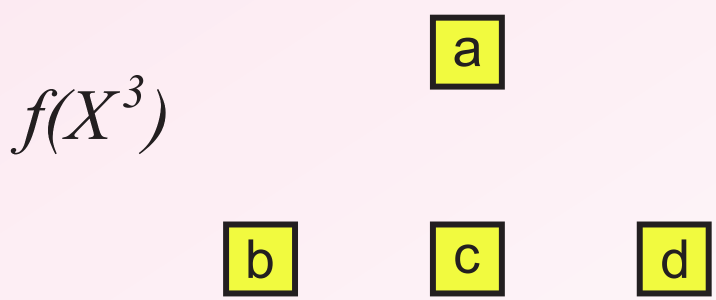

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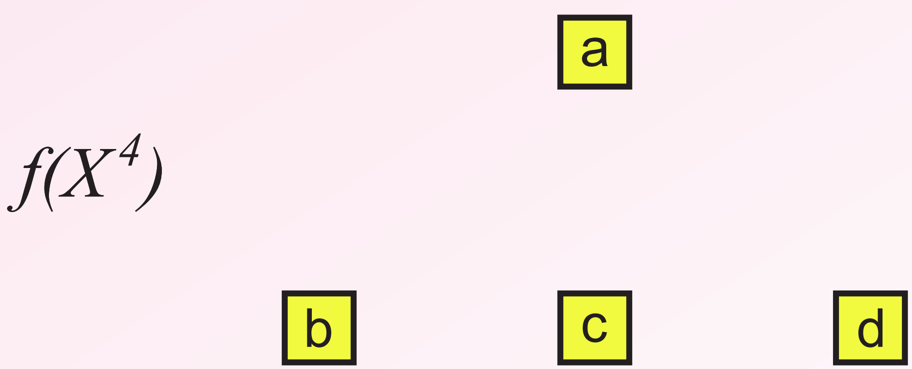

Lemma 1: There is an integer M such that $f(X^n) \in \mathrm{Ann}(c)$ for every $n \in \mathbb{N}$ that is relatively prime with M.

Proof: First consider the case that $n = p$ is a prime. Then

$$f(X)^p \equiv f(X^p) \pmod{p\mathbb{Z}[X]}$$

so because $f(X)c(X) = 0$ we have

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The coefficients in $f(X^p)c(X)$ are bounded in absolute value by

$$m = \sum_{I \in \mathcal{I}} |a_I| c_{\mathrm{max}}$$

where c_{max} is the maximum absolute value of the coefficients in c.

If $p > m$ then $f(X^p)c(X) = 0$. We choose $M = m!$ so any number relatively prime with M is $> m$.

Lemma 1: There is an integer M such that $f(X^n) \in \text{Ann}(c)$ for every $n \in \mathbb{N}$ that is relatively prime with M.

Proof: General case $n = p_1 \dots p_k$ where primes p_k are relatively prime with M.

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Proof: General case $n = p_1 \dots p_k$ where primes p_k are relatively prime with M.

Using the prime case we have

$$\begin{aligned} f(X^{p_1}) \in \text{Ann}(c) \text{ so} \\ f(X^{p_1 p_2}) \in \text{Ann}(c) \text{ so} \\ \dots \\ f(X^{p_1 p_2 \dots p_k}) \in \text{Ann}(c). \end{aligned}$$

Note that the coefficients of all $f(X^n)$ are the same as in $f(X)$ so the same M can be used everywhere. $\square$

Lemma 1: There is an integer M such that $f(X^n) \in \mathrm{Ann}(c)$ for every $n \in \mathbb{N}$ that is relatively prime with M.

In particular, $f(X^{1+iM})$ are in $\mathrm{Ann}(c)$ for $i = 0, 1, 2, \ldots$.

Let $Z \in \mathbb{C}^d$ be a common zero of $\mathrm{Ann}(c)$. Then

$$f(Z^{1+iM}) = 0 \text{ for all } i = 0, 1, 2, \ldots$$

In the following we show that $g(Z) = 0$ where

$$g(X) = X^1 \prod_{\substack{I,J \in \mathcal{I} \\ I \neq J}} (X^{MI} - X^{MJ}).$$

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Case 1: Some coordinate z_i of $Z = (z_1, \ldots, z_d)$ is zero.

Then $Z^{(1,\ldots,1)} = 0$. Because Z is a zero of the monomial X^1,

$$g(Z) = 0.$$

Let $Z \in \mathbb{C}^d$ be a common zero of $\mathrm{Ann}(c)$. Then

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Case 2: Assume then that all coordinates $z_i \neq 0$.

We partition the monomials of $f(X)$ according to the value of X^I for $X = Z^M$. For any $\alpha \in \mathbb{C}$ denote

$$\mathcal{I}_\alpha = \{I \in \mathcal{I} \mid Z^{MI} = \alpha\}.$$

Then $\mathcal{I}$ is partitioned into some $\mathcal{I}_{\alpha_1}, \ldots, \mathcal{I}_{\alpha_n}$.

$$\begin{aligned} f(X) &= \sum_{I\in\mathcal{I}} a_I X^I \\ f(Z^{1+iM}) &= 0, \qquad \text{for all } i = 0, 1, 2, \ldots \\ \mathcal{I}_\alpha &= \{I \in \mathcal{I} \mid Z^{MI} = \alpha\} \end{aligned}$$

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\hline
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Denote

$$f_\alpha(X) = \sum_{I\in\mathcal{I}_\alpha} a_I X^I.$$

Then

$$f(X) = f_{\alpha_1}(X) + \cdots + f_{\alpha_n}(X).$$

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In other words, for all i vectors

$(f_{\alpha_1}(Z), \ldots, f_{\alpha_n}(Z))$ and $(\alpha_1^i, \alpha_2^i, \ldots, \alpha_n^i)$

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are orthogonal. This implies that all $f_{\alpha_i}(Z) = 0$. $\square$

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Because all coordinates z_i of Z are non-zero, $Z^I \neq 0$ for all I.

If some $\mathcal{I}_\alpha$ would contain exactly one $I \in \mathcal{I}$ then

$$0 = f_\alpha(Z) = a_I Z^I \neq 0.$$

So any non-empty $\mathcal{I}_\alpha$ contains at least two distinct $I, J \in \mathcal{I}$.

But $Z^{MI} = \alpha = Z^{MJ}$, so Z is a zero of $X^{MI} - X^{MJ}$.

We have proved the following: All elements of the variety

$$\{Z \in \mathbb{C}^d \mid \ f(Z) = 0 \text{ for all } f \in \mathrm{Ann}(c) \ \}$$

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Nullstellensatz $\Longrightarrow g(X)^n \in \mathrm{Ann}(c)$ for some $n \in \mathbb{N}$.

Dividing $g(X)^n$ by a suitable monomial gives:

Theorem. Finitary, integral $c(X)$ that has a non-trivial annihilator is annihilated by a Laurent polynomial of the form

$$(1 - X^{I_1})(1 - X^{I_2}) \dots (1 - X^{I_k}).$$

Annihilator: $(1 - X^{I_1})(1 - X^{I_2}) \dots (1 - X^{I_k})$

Binomials $(1 - X^I)$ correspond to **difference operators** that subtract from a configuration its own I-translation.

The theorem states that configuration $c(X)$ can be annihilated by a sequence of difference operations.

$$\textbf{Annihilator:} \quad (1 - X^{I_1})(1 - X^{I_2}) \ldots (1 - X^{I_k})$$

Binomials $(1 - X^I)$ correspond to **difference operators** that subtract from a configuration its own I-translation.

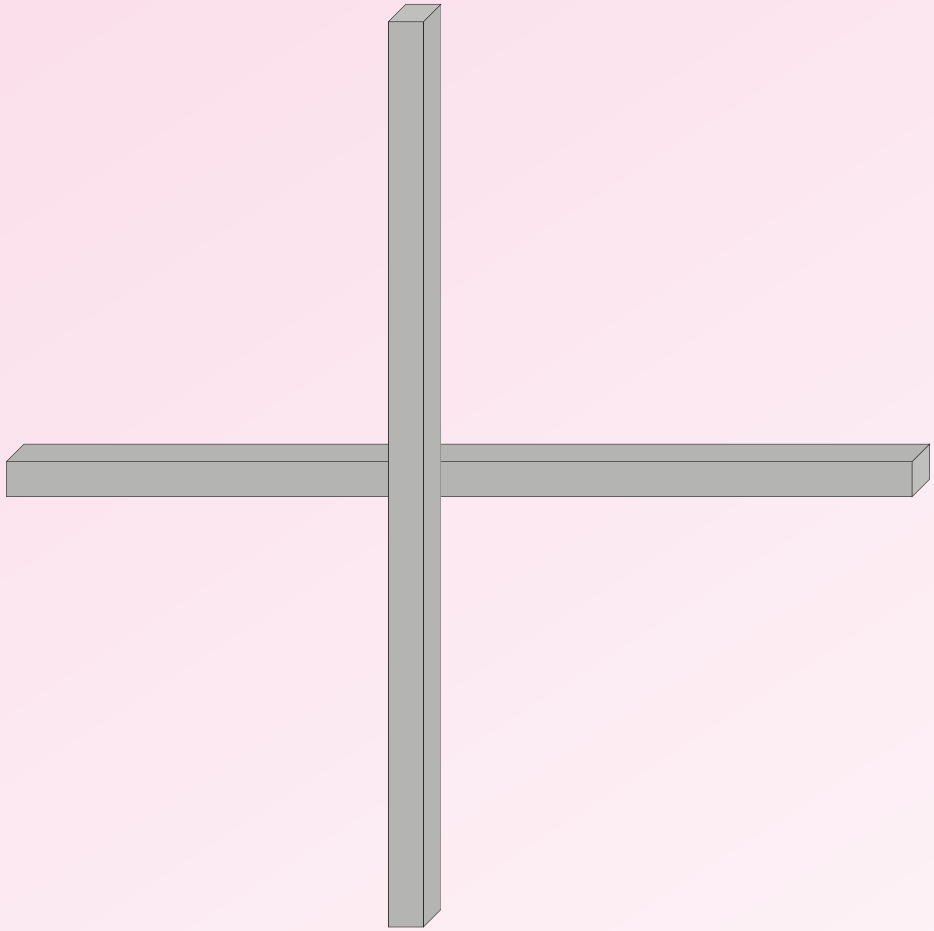
The theorem states that configuration $c(X)$ can be annihilated by a sequence of difference operations.

If $k = 1$ then $c(X)$ is periodic.

More generally, we can prove that $c(X)$ is a sum of k (possibly non-finitary) integral configurations that are periodic.

Corollary. $c(X) = c_1(X) + \cdots + c_k(X)$ where $c_i(X)$ is I_i-periodic and integral (but not necessarily finitary).

Example. The 3D counter example



to Nivat's conjecture is a sum of two periodic configurations. It is annihilated by polynomial $(1-y)(1-x)$.

Our approach to Nivat's conjecture.

Suppose $P(c, D) \leq |D|$ for some rectangle D.

Then c has annihilating polynomial

$$f(X) = (1 - X^{I_1}) \dots (1 - X^{I_k}).$$

Take the one with smallest k.

If $k = 1$ then c is periodic, so assume that $k \geq 2$.

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Take the one with smallest k.

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Denote $\delta_i(X) = (1 - X^{I_i})$ and $\phi_i(X) = f(X)/\delta_i(X)$.

Then $\phi_i(X)c(X)$ is annihilated by $\delta_i(X)$ so it is I_i-periodic. It is not doubly periodic (since otherwise k could be reduced).

Viewing $c(X)$ using filter $\phi_1(X)$:

Non-periodic sequence of stripes in the direction I_1.

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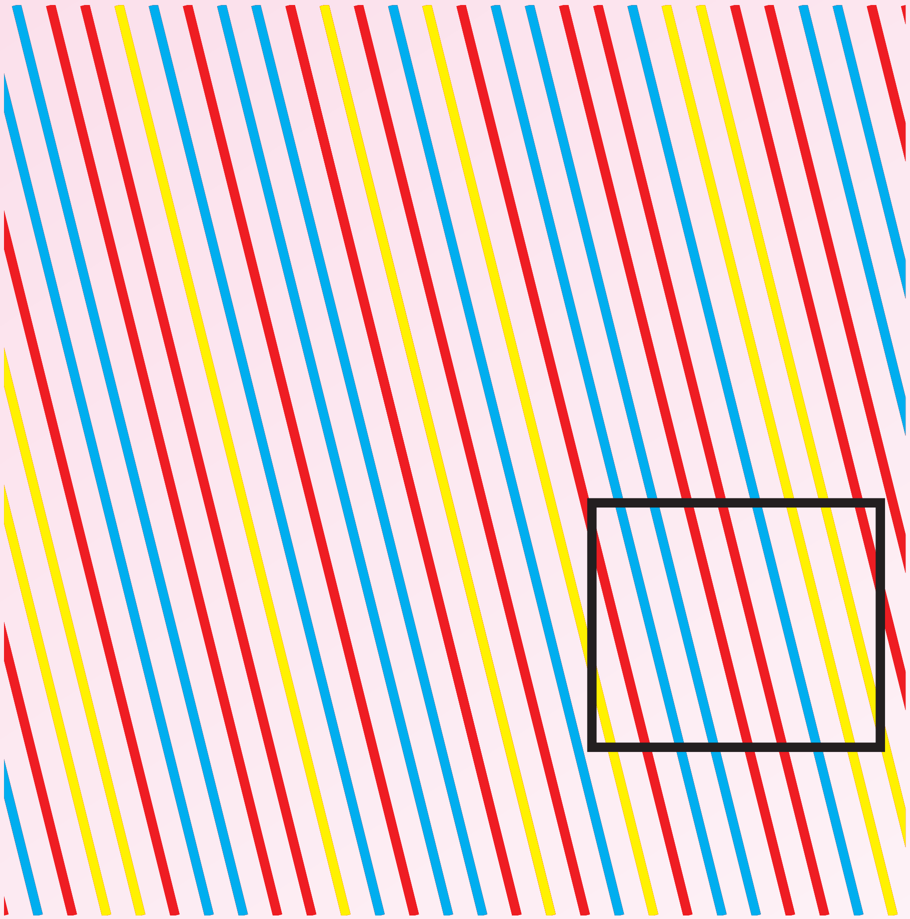
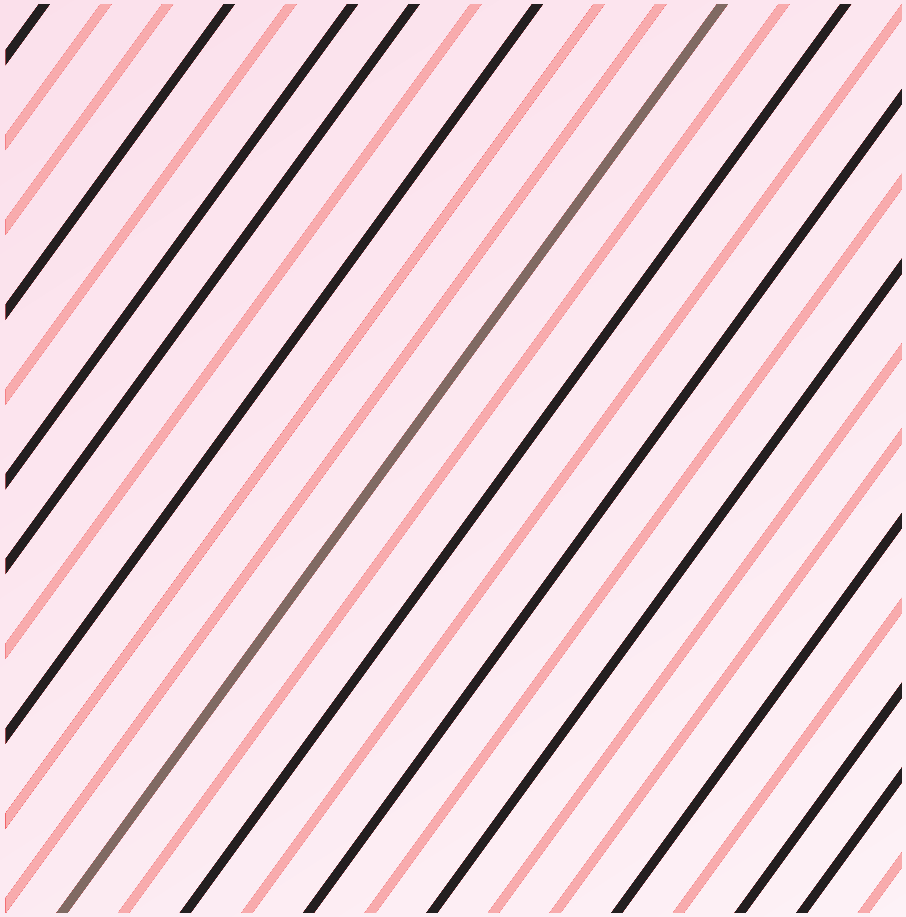


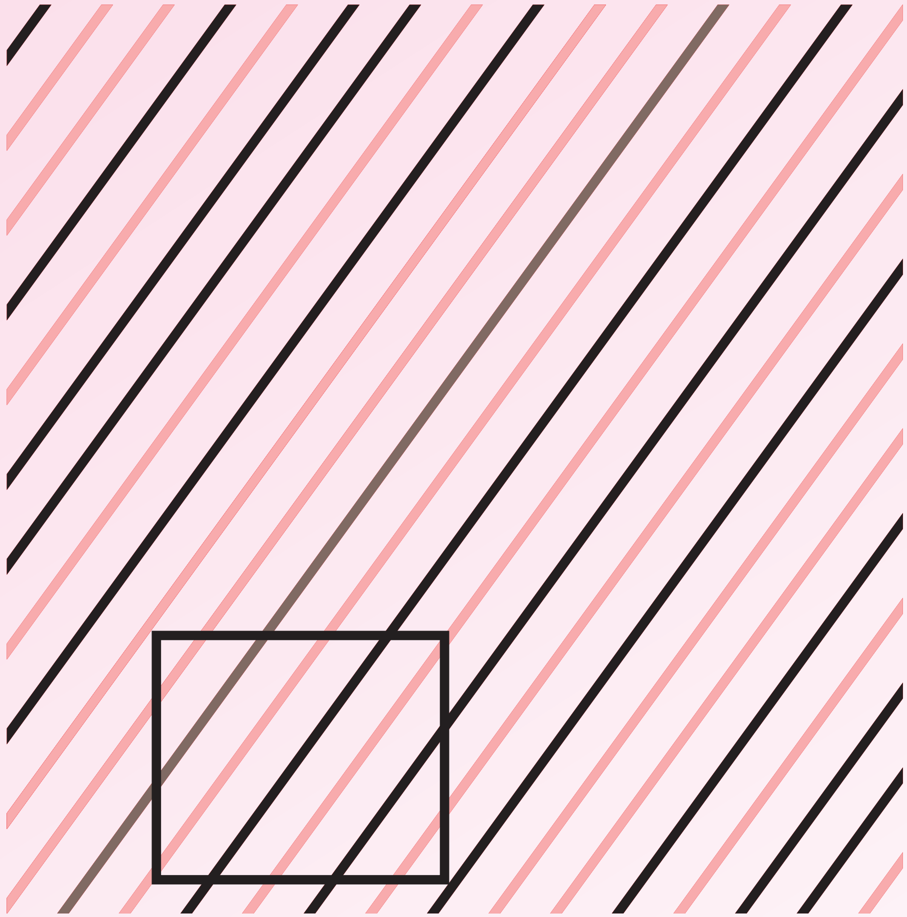
If at least u stripes are visible in every $X \times Y$ rectangle then (by the one-dimensional Morse-Hedlund theorem) there are more than u different $X \times Y$ blocks in $\phi_1(X)c(X)$.

Viewing $c(X)$ using filter $\phi_2(X)$:



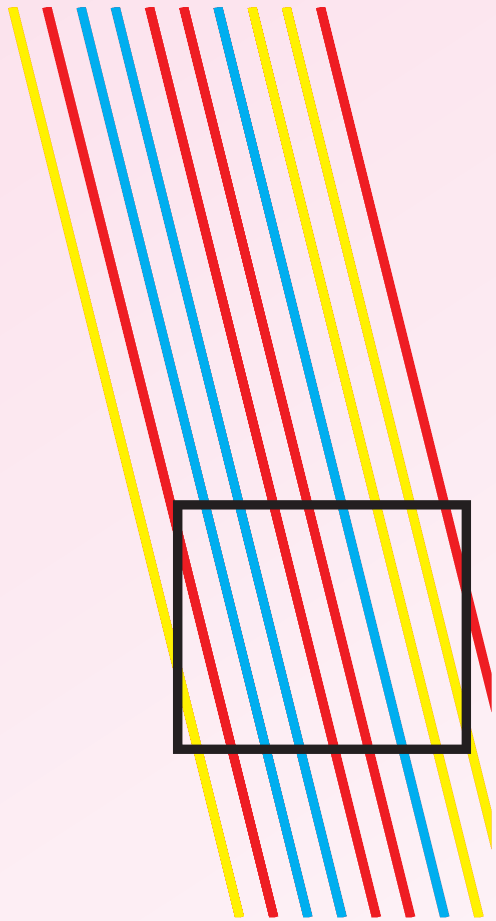
Non-periodic sequence of stripes in a different direction I_2.

Viewing $c(X)$ using filter $\phi_2(X)$:

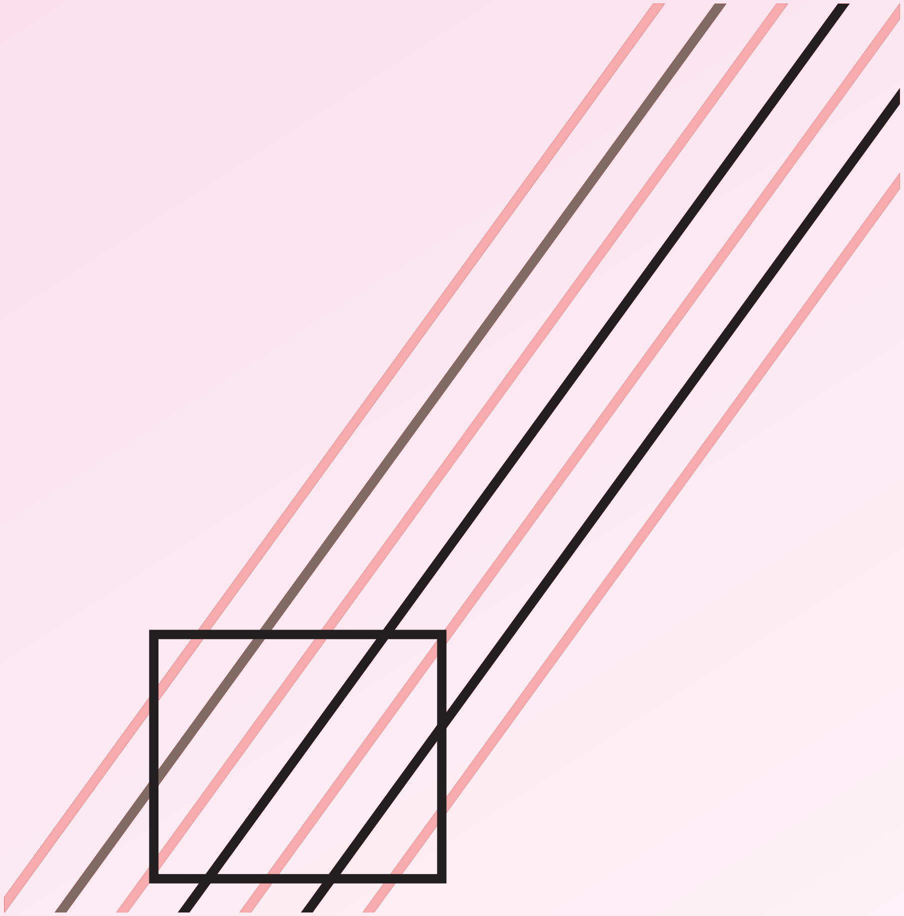


At least v stripes visible in $X \times Y$ rectangles $\Longrightarrow$ more than v different $X \times Y$ blocks in $\phi_2(X)c(X)$.

Pick any $X \times Y$ pattern from $\phi_1(X)c(X)\ldots$



…and any $X \times Y$ pattern from $\phi_2(X)c(X)$.



Directions I_1 and I_2 are different

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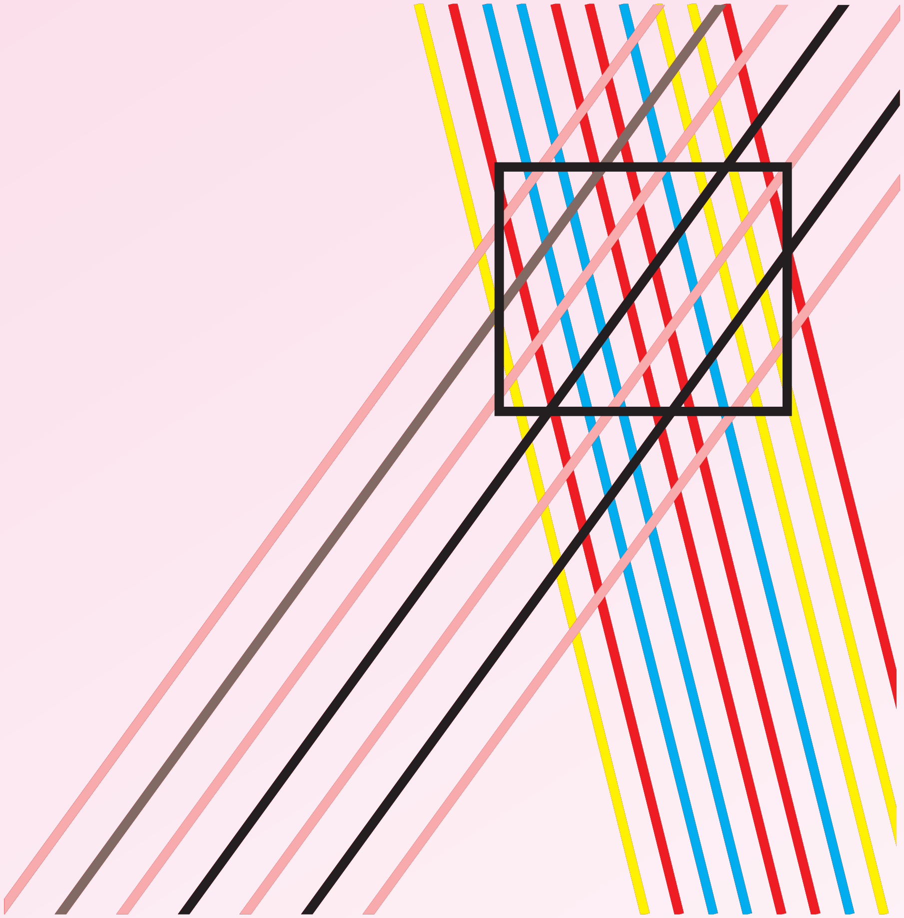

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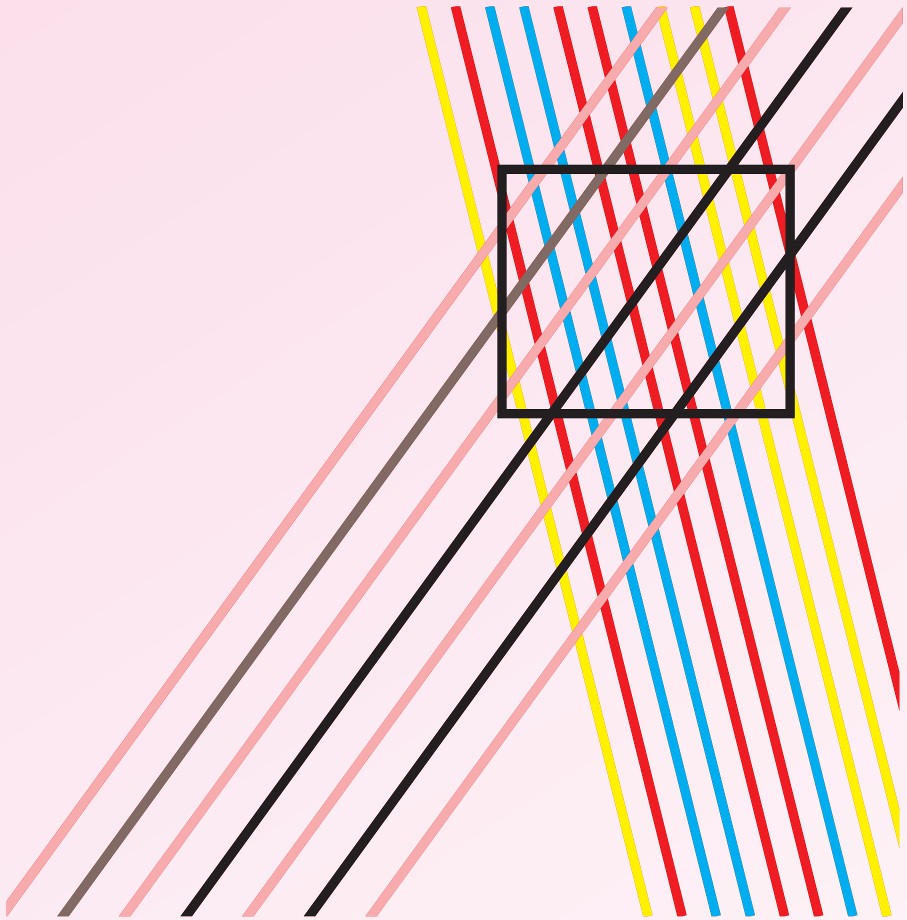

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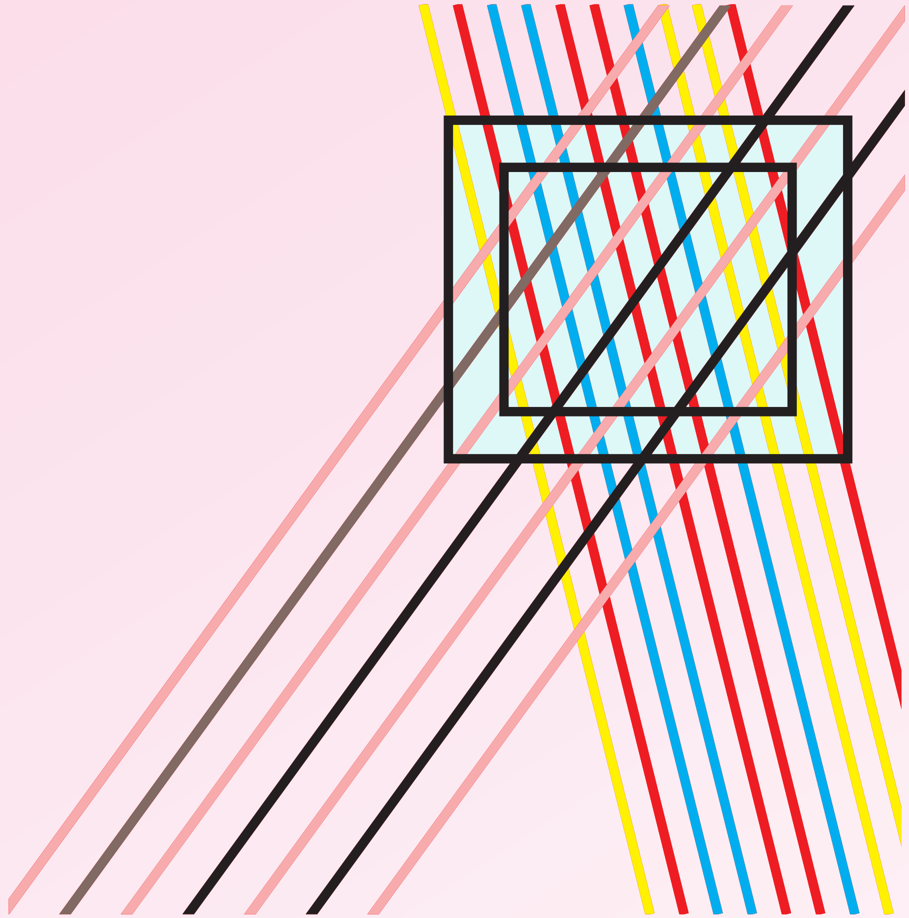
$\Longrightarrow uv \geq XY$ different pairs of patterns



For some constant r (=radius of filters ϕ_1 and ϕ_2), each $(X+2r)\times(Y+2r)$ block of $c(X)$ uniquely determines the corresponding $X\times Y$ blocks in $\phi_1(X)c(X)$ and $\phi_2(X)c(X)$.

$\Longrightarrow c(X)$ has at least XY patterns of size $(X+2r)\times(Y+2r)$.

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$$\liminf_{D} \frac{P(c, D)}{|D|} \geq 1.$$

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This can be improved further to:

Theorem. If c is a non-periodic 2D configuration then $P(c,D) \leq |D|$ can hold only for finitely many rectangles D.

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Thank You