Automorphisms of symbolic dynamical systems

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Isomorphism, factor maps, subsystems, subactions

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- If $\alpha \in \mathrm{Hom}(\mathbb{H}, \mathbb{G})$ is a group homomorphism and (X, T) is a $\mathbb{G}$-flow, we can get an $\mathbb{H}$-flow $(X, T^{(\alpha)})$ by $T^{(\alpha)}_h := T_{\alpha(h)}$. When α is injective, we say $(X, T^{(\alpha)})$ is a subaction of (X, T).

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- The collection of ordered perfect matchings in $\mathcal{G}_S$ can be viewed as an SFT ("dimer models").

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- Embeds $Aut(\{1, \dots, n\}^{\mathbb{Z}}, \sigma)$ for all n [KR].

On the automorphism group of $\mathbb{Z}$-SFTs

[BLR], extending [H] Let (X, σ) be an uncountable $\mathbb{Z}$-SFT. Then $Aut(X, \sigma)$:

- Embeds any finite group.
- Embeds the direct sum of every countable collection of finite groups.
- Embeds a free product of any number of 2-element groups, hence a free group on a countable number of generators.
- Embeds $Aut(\{1, \ldots, n\}^{\mathbb{Z}}, \sigma)$ for all n [KR].
- If (X, σ) is irreducible then the center or $Aut(X, \sigma)$ generated by σ [R].

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Let's prove that $Aut(X,\sigma)$ embeds the free product of cyclic groups:

- For simplicity, let $X = \{*, 0, 1, 2, 3\}^{\mathbb{Z}}$.
- For $j \in \{1,2,3\}$ define $\phi_j \in Aut(X,\sigma)$ to be the automorphism which "swaps sj with $s0$" for $s \notin \{0, j\}$.

Embedding free products:

Let's prove that $Aut(X, \sigma)$ embeds the free product of cyclic groups:

- For simplicity, let $X = \{*, 0, 1, 2, 3\}^{\mathbb{Z}}$.
- For $j \in \{1, 2, 3\}$ define $\phi_j \in Aut(X, \sigma)$ to be the automorphism which "swaps sj with $s0$" for $s \notin \{0, j\}$.
- By inspecting the action of $\langle \phi_1, \phi_2, \phi_3 \rangle$ on the point $\ldots 000 * 000 \ldots$ we see that it generates a group isomorphic to the free product $C_2 * C_2 * C_2$.

Coming up next...

- Topological Markov Fields.
- Amenability and topological entropy.
- The Marker method for $\mathbb{Z}^d$-SFTs and for $\mathbb{G}$-SFTs.